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Ярославский государственный университет им. П.Г. Демидова

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Yaroslavl, October 7 - 11, 2024

P. G. Demidov Yaroslavl State University
Regional Scientific and Educational Mathematical Center
"Centre of Integrable Systems"
Steklov Mathematical Institute of the Russian Academy of Sciences, Moscow
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Abstracts

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MULTISOLITON KP-2 SOLUTIONS AND DEGENERATE M-CURVES

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Multiline solitons of the KP-2 equations can be constructed using the Darboux transformations. But they also can be obtained as degeneration of the finite-gap solutions. If one wants to obtain physically relevant regular real-valued solutions it is necessary to use totally-positive Grassmannians in the Darboux transformation and M-curves in the finite-gap approach. We construct a bridge between these two approaches.

POLYNOMIAL INTEGRALS OF GEODESIC FLOWS AND THE GENERALIZED HODOGRAPH METHOD

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We consider integrable geodesic flows on 2-surfaces admitting an additional polynomial in momenta first integral. Generally speaking, the search for such integrals leads to certain complicated quasi-linear systems of PDEs. As proved in [1], typically these systems turned out to be *semi-Hamiltonian* ([2]). In particular, this allows to apply *the generalized hodograph method* ([2]) to construct solutions to such systems. However, the direct implementation of this method turned out to be an implicit and very complicated procedure ([3], [4], see also [5]).

We present an explicit algorithm based on the generalized hodograph method which allows one to construct many particular solutions to these systems. Explicit examples of metrics and first integrals of high degrees are also provided.

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TWO-CLUSTER SYNCHRONISATION IN A FULLY COUPLED NETWORK OF MACKEY–GLASS GENERATORS

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The Mackey – Glass generator is an electric generator whose operation is described by the Mackey – Glass equation [1]

$$\frac{dV}{dt} = -bV + \frac{acV(t-\tau)}{1+(cV(t-\tau))^{\gamma}}$$

Here, $V(t)$ is the voltage function, $a > 0$ is the saturation level of nonlinearity, $b > 0$ is the RC constant, $\tau > 0$ is the time delay, the parameter $\gamma > 0$ determines the shape of the nonlinear function, and $c > 0$ is the feedback strength.

Fix $m, n \in \mathbb{N}$. Consider a fully coupled network of $N = m + n$ Mackey-Glass generators, i. e. a network, where each generator is coupled to each. This network described by the equation

$$\frac{dV_j}{dt} = -bV_j + \frac{ac\left(V_j(t-\tau) + \delta\sum_{k=1,k\neq j}^{N} V_k(t)\right)}{1+\left(c\left(V_j(t-\tau) + \delta\sum_{k=1,k\neq j}^{N} V_k(t)\right)\right)^{\gamma}}, \quad j = 1, 2, \ldots, N,$$

where the parameter $\delta > 1$ controlling the strength of the coupling.

After substitutions $V_j = c^{-1}u_j(\frac{t}{\tau})$, $\beta = b\tau$, $\alpha = ac\tau$, $t \mapsto \frac{t}{\tau}$, we obtain

$$\dot{u}_j = -\beta u_j + \frac{\alpha\big(u_j(t-1) + \delta\sum_{k=1,k\neq j}^{N} u_k(t)\big)}{1 + \big(u_j(t-1) + \delta\sum_{k=1,k\neq j}^{N} u_k(t)\big)^{\gamma}}, \quad j = 1, 2, \ldots, N. \tag{1}$$

Here $\alpha, \beta > 0$.

Let

$$F(x) = \frac{x}{1 + x^{\gamma}}. \tag{2}$$

Considering (2), the system (1) takes the form

$$\dot{u}_j = -\beta u_j + \alpha F\left(u_j(t-1) + \delta \sum_{k=1,k\neq j}^{N} u_k(t)\right), \quad j = 1, \ldots, N.$$

We will look for two-cluster synchronization modes, i.e. periodic modes in which m generators are described by the function $u(t)$, and the remaining n generators by the function $v(t)$.

In this consideration, the system (1) takes the form

$$\begin{cases} \dot{u} = -\beta u + \alpha F\big(u(t-1) + \delta(m-1)u + \delta n v\big), \\ \dot{v} = -\beta v + \alpha F\big(v(t-1) + \delta m u + \delta(n-1)v\big). \end{cases} \tag{3}$$

In this work, we seek a stable periodic solution of the system (3).

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NOTES ON THE APPROACH, METHODS AND RESULTS OF A QUALITATIVE RESEARCH FOR SOME FAMILIES OF POLYNOMIAL DYNAMIC SYSTEMS

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A key role in multiple branches of contemporary mathematical modeling belongs to dynamic systems. The examples of such an approach can be found in various directions of science and engineering, such as mathematical models of astrophysical, geophysical and (especially) atmospheric processes; the wide spectrum of tasks of engineering, for example, the problems of seismic stability; basic analysis of computing and producing systems; studies in sociological and ecological processes.

A proper dynamic system serves as a mathematical apparatus during a research of some phenomena and conditions, which permit to ignore any statistical events. The point is to study curves, defined by differential equations of a taken dynamic system.

Conducting such an analysis, firstly we subdivide the phase space into separate trajectories.

Next, we investigate a limit behavior of those trajectories with the aim to find and to construct the classification of possible equilibrium positions. Also, at this research stage we find out possible sinks and sources of the phase flow. After all these steps, we construct a global set of possible phase portraits, which a given differential dynamic system may have, and this mean that we can describe and predict the development of a physical process under investigation.

Polynomial dynamic systems play especially important role as practical mathematical models, and it is a reason for their preferential in-depth study.

Our talk describes the mathematical tasks, special methodology, progress and results of the thorough original study of a wide hierarchical family of differential polynomial dynamic systems with reciprocal right parts, having broad application prospects.

The considered family of cubic systems have the form

$$\begin{aligned} \frac{dx}{dt} &= p_0x^3 + p_1x^2y + p_2xy^2 + p_3y^3 \equiv X(x,y), \\ \frac{dy}{dt} &= ax^2 + bxy + cy^2 \equiv Y(x,y), \end{aligned} \tag{1}$$

where a, b, c, p_0, …, p_3 — are the real parameters, and for them : $c\,p_3 \neq 0$, while X, Y — are reciprocal polynomials.

The goal of our study is to reveal and depict the whole set of topologically different phase portraits in the enclosed Poincare disk $\overline{\Omega}$, which can be realized for the (having some complicated hierarchy of multiple subfamilies) dynamic systems of the family (1), and find conditions of their existence.

Using the classical approach of the qualitative theory of ODEs, i.e. the first and the second Poincare transformations: the central mapping at first and the orthogonal mapping secondly, together with several new notations and methods, especially developed for the aims of the present investigation, we subsequently study by a common plan all the existing subfamilies of the family (1), which appeared to belong to several hierarchical levels. The actual number of those levels varies from 3 to 4 for separate edges of the hierarchy graph.

As a result of the conducted work more than 250 topologically different phase portraits were depicted in the Poincare disk. Criteria of their existence were given. There was proved the total absence of limit cycles for the whole global (1)-family of polynomial dynamic systems.

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MATHEMATICAL MODELING OF A MOVING FIRE FRONT IN SPREADING FOREST FIRES

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This work is devoted to the study of mathematical models of landscape forest fires and numerical modeling of the movement of the forest fire front. The problems of modeling forest fires were previously considered, for example, in studies [1-3]. In our work, the main object of study is the problem of determining the position of the moving front of a forest fire. It is known that as a result of forest fires, not all forest biomass burns, but only a part of it [4]. Knowing information about the movement of the fire front, it is possible not only to predict the further direction of fire spread, but also to estimate the damage (the proportion of burnt biomass) caused to the forest in the area where the forest fire has already passed [5].

The forest fire model considered in this paper consists of two equations: an advection-diffusion type equation for temperature and a reaction-diffusion type equation for biomass. The right-hand side of the equations under study may contain cubic nonlinearity or discontinuity. The visible front of forest fire depends on the size of the observation area. In the case of a relatively small observation altitude (e.g., from an airplane or unmanned aerial vehicle), the visible front is an open curve. We will call such a forest fire problem formulation a local formulation. In the case of forest fire observation from a high altitude (e.g., using satellite observation data), the visible front of fire will be a closed curve, and the corresponding problem will be considered in a global formulation.

A numerical simulation of the forest fire front movement was performed using finite-difference methods. Examples are given for different environmental parameters and wind directions, and a number of a posteriori accuracy estimates are made based on numerical experiments for one of the problems from the class under consideration.

The work was supported by the Foundation for the Development of Theoretical Physics and Mathematics “BASIS”.

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LINEARIZATION OF DELAY DIFFERENTIAL EQUATIONS WITH DELAYED DISCONTINUITIES

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Consider the differential equation of the form

$$\dot{x} = f(t, x, x_\tau) = \begin{cases} f_+(t, x, x_\tau), & b(x_\tau) > 0, \\ f_-(t, x, x_\tau), & b(x_\tau) < 0, \end{cases} \tag{1}$$

where $x = x(t) \in \mathbb{R}^n$; $x_\tau = x_\tau(t) = x(t - \tau(t))$; time-dependant delay $\tau(t) > 0$ is C^1, bounded and non-vanishing; $b : \mathbb{R}^n \to \mathbb{R}$ is a scalar

C^1-function; functions $f_\pm$ are C^1 on the sets $\mathbb{R}\times\mathbb{R}^n\times\{x_\tau\in\mathbb{R}^n : \pm b(x_\tau)\geqslant 0\}$ respectively.

Since generally the right hand side of (1) is discontinuous, instead of classical solutions we consider Carathéodory solutions of (1).

Variational equation describes the evolution of infinitesimal perturbations of initial conditions over time, playing extremely important role in study of dynamical systems with regard to stability (first Lyapunov's method), hyperbolicity, and Lyapunov exponents, the theory of which is well developed for continuous systems. Systems of the type (1) are not uncommon, they appear as limit objects for systems with large parameter, in relay control systems, or by themself as models for real-life phenomena. Hence, generalizing existing tools for study of dynamical systems of the type (1) is very important.

Definition 1. The solution $x(t)$ of (1) is called **non-degenerate** if it's domain is $[t_0-\overline{\tau},+\infty)$ for some t_0 and

1. The set $Z_x=\big\{T\ :\ b(x_\tau(T))=0\big\}$ consists of isolated points. In other words, $x(t)$ is not a sliding mode.
2. For all $T\in Z_x$, function $t\mapsto b(x_\tau(t))$ is differentiable at $t=T$ and it's derivative at $t=T$ is not zero. In other words, the retarded solution $x_\tau(t)$ crosses discontinuity surface $b=0$ smoothly and transversally.

Definition 2. Let $x(t)$ be a non-degenerate solution of (1) defined on $[t_0-\overline{\tau},+\infty)$. Consider an initial value problem

$$\begin{cases}\dot w=\partial_x f\cdot w+\partial_{x_\tau}f\cdot w_\tau+\displaystyle\sum_{T\in Z_x}\delta(t-T)\frac{(f_+-f_-)b'(x_\tau)w_\tau}{|(1-\tau')b'(x_\tau)\dot x_\tau|},\\ w(t)=\psi(t),\quad t\in[t_0-\overline{\tau},t_0]\end{cases}\tag{2}$$

where f, $\partial_x f$ and $\partial_{x_\tau}f$ are evaluated at (t,x,x_τ); δ is Dirac delta function; $\dot x_\tau=\dfrac{dx}{dt}(t-\tau(t))$. The differential equation in (2) is called a **variational equation** (or **linearization**) for equation (1) at it's solution $x(t)$.

Theorem 1. *Let $x(t)=x(t,\varphi)$ and $y(t)=x(t,\varphi+\varepsilon\psi)$ be the solutions of (1) satisfying initial conditions*

$$x(t)=\varphi(t),\quad y(t)=\varphi(t)+\varepsilon\psi(t)$$

for $t\in[t_0-\overline{\tau},t_0]$ respectively, ψ being the infinitesimal perturbation of initial conditions as $\varepsilon\to 0$. Suppose that $x(t)$ is non-degenerate *solution. Let $\varepsilon w_\varepsilon = y(t) - x(t)$ be the evolution of $\varepsilon\psi$-perturbation of initial conditions, then $w_\varepsilon \to w$ as $\varepsilon \to 0$ pointwise for any $t \notin Z_x$ and uniformly on any bounded interval minus any open set containing Z_x.*

BIFURCATIONS OF SMALE HORSESHOES' CHAINS IN A DOUBLE-SCROLL ATTRACTOR

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In this talk we consider bifurcations of piecewise-smooth system A composed from three linear subsystems A_0, A_l and A_r :

$$A_0: \quad \begin{array}{l} \dot{x} = x, \\ \dot{y} = -\nu y + \omega z, \\ \dot{z} = -\omega y - \nu z, \end{array} \qquad \text{for} \quad (x, y, z) \in G_0,$$

$$A_l: \quad \begin{array}{l} \dot{x} = -\alpha(x + h) - \Omega(z + 1), \\ \dot{y} = -\beta y, \\ \dot{z} = \Omega(x + h) - \alpha(z + 1), \end{array} \qquad \text{for} \quad (x, y, z) \in G_l,$$

$$A_r: \quad \begin{array}{l} \dot{x} = -\alpha(x - h) - \Omega(z - 1), \\ \dot{y} = -\beta y, \\ \dot{z} = \Omega(x - h) - \alpha(z - 1), \end{array} \qquad \text{for} \quad (x, y, z) \in G_r,$$

where h, α, β, ν, ω and Ω are positive parameters, and regions G_0, G_l and G_r are defined as follows

$$\begin{array}{lcl} G_0 & : & |x| < h, (y^2 + z^2 \leq r^2) \cap (|z| < 1), \\ G_l & : & (z \leq -\operatorname{sign} x, y \in \mathbb{R}^1) \setminus G_0, \\ G_r & : & (z \geq -\operatorname{sign} x, y \in \mathbb{R}^1) \setminus G_0, \end{array}$$

for some positive parameter $r > 1$.

This system was introduced in our recent paper [1] as a certain dynamical system with a double homoclinic loop to saddle-focus allowing its complete analytical study. For this system we analytically obtained Poincaré return map in explicit form and proved the existence of a double-scroll attractor. In order to get a complete description of the attractor structure we introduce so-called *Smale horseshoe chains*, which made it possible to reveal the true complexity of the double-scroll attractor.

It this talk, we continue to use the analytical advantages of system A and consider the issue of bifurcations of invariant sets, belonging to the attractor and defined by Smale horseshoes chains. In particular, we show that the infinitesimal increase of bifurcation parameter μ from zero leads to birth and disappear of countable set of double-scroll attractors and spiral attractors.

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ON TRANSCENDENTAL CASES IN THE PROBLEM OF ORBITAL STABILITY OF PERIODIC MOTIONS OF A HEAVY RIGID BODY

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We deal with the problem of orbital stability of the pendulum-like oscillations of a heavy rigid body with a fixed point. It is supposed that the mass geometry of the body corresponds to the Hess case [1]. In this case the system of perturbed motion equations has three parameters. It was shown that the first order resonance takes place for all values of parameters in this system [2], i.e. the characteristic equation of the linearized system has double root, which is equal to 1. In such a resonant case the study of the linearized system is not enough to obtain conclusions on orbital stability of the pendulum-like oscillations.

In this work we write down the system of equations describing the perturbed motion in a neighborhood of pendulum-like oscillations in a Hamiltonian form and perform rigorous nonlinear stability analysis. We show that the orbital stability problem cannot be solved by means of the study of a non-linear approximation of any finite order, i.e. we show that the so-called transcendental situation [3] takes place in the considered problem. In general position, when the monodromy matrix is non-diagonalizable, by using the results of paper [4] we prove the orbital instability of the pendulum-like oscillations. We also consider the special case, when monodromy matrix can be brought into a diagonal form and the pendulum-like oscillations are stable in the linear approximation. This case takes place on a two-dimensional surface in three-dimensional space of parameters. The calculations have shown that on this surface the normal form of the Hamiltonian of the perturbed motion does not includes terms of degree two and three. By means of the method proposed in paper [4] we proved that in such a special case the corresponding Hamiltonian system is unstable. Thus, in spite of the orbital stability in the linear approximation the pendulum-like oscillations are orbitally unstable in the original non-linear system.

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STUDY OF MULTIDIMENSIONAL MAPS USING ONE-DIMENSIONAL ENDOMORPHISMS: SMALL PARAMETER APPROACH

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In recent paper [1] it had been proved that the bifurcation structure of a quadratic nonivertible map persists when the parameter increases from zero and the map turns into an invertible multidimensional Henon map. In this talk we discuss the similar problem for a generalized map which combines the Henon type maps, the Poincare return map for Shilnikov bifurcation of saddle-focus homoclinic orbit, the Lurie discrete time system, etc. [2-6].

We have obtained the expected result about the persistence of periodic orbits and their bifurcations when passing from a one-dimensional endomorphism to the generalized map when a small parameter becomes non-zero.

We have revealed the precise mechanism of change of homoclinic orbits and splitting of unstable manifolds as a result of the transition of 1-D endomorphism to multidimensional map. Thereby we have derived the reconstruction rules of nonwandering set of orbits and bifurcations of the generalized map from those of 1-D endomorphism.

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A NONSTANDARD BILLIARD PROBLEM

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Let (M, g) be a Riemannian manifold, $\Omega \subset M$ a domain with smooth boundary Γ, and ϕ be a smooth function such that $\phi|_\Omega > 0$, $\phi|_\Gamma = 0$, and $d\phi|_\Gamma \neq 0$. We study the geodesic flow of the metric $G = g/\phi$ in Ω. The G-distance from any point of Ω to Γ is finite, so the geodesic flow is incomplete. Regularization of the flow in a neighborhood of Γ establishes a natural reflection law from Γ. This leads to a certain billiard-like problem in Ω. We obtain a normal form for the regularized flow near Γ and for the corresponding billiard map of $T^*\Gamma$. This leads to a version of Lazutkin's theorem [1] on the existence of caustics for convex billiards. Our work was motivated by the results of Dobrohotov and Nazaikinkii, see e.g. [2], on the quasi-classical approximation for the wave equation $u_{tt} = \nabla \cdot (\phi \nabla u)$ in Ω degenerating on Γ. The talk is based on the paper [3].

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SOLVABILITY THEOREMS FOR THE LINEAR NON-LOCAL PROBLEM FOR ABSTRACT PARABOLIC EQUATIONS

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Consider a triple $V \subset H \subset V'$ of separable Hilbert spaces, where V' is the dual space of V and H is identified with its dual H'. Both embeddings are dense and continuous. Let $a(u, v)$ be a sesquilinear form defined on the space V and satisfying the estimates

$$|a(u, v)| \leqslant \mu \|u\|_V \|v\|_V, \qquad \operatorname{Re} a(u, u) \geqslant \alpha \|u\|_V^2, \tag{1}$$

for all $u, v \in V$, where μ and α are positive constants. Obviously, the form $a(u, v)$ generates a bounded linear operator $A : V \to V'$ such that $a(u, v) = (Au, v)$ for all $u, v \in V$. This implies the estimate $\|A\|_{V \to V'} \leqslant \mu$. We denote by (z, v) the value of a functional $z \in V'$ on an element $v \in V$. Due to the identification $H \equiv H'$, the expression (z, v) coincides with the scalar product in H for $z \in H$ [1].

In the space V' we consider the parabolic problem

$$u'(t) + Au(t) = f(t), \qquad \int_0^T p(t) u'(t)\, dt = \overline{u}\,. \tag{2}$$

on the interval $[0, T]$. In (2), the function $t \to f(t) \in V'$, the element $\overline{u}$ and the function $t \to p(t) \in \mathbb{R}^1$ are given. Here and below, all derivatives are meant in the generalized sense.

We refer to the article [2] close in the subject area, where solvability of an abstract parabolic problem with the operator $A(t)$ dependent on time and the periodic condition is studied. In [3] and [4], weak and smooth solvability theorems of a parabolic equation with a weighted integral condition are proved. In [5] the criterion of uniqueness of the solution to a parabolic problem with a non-local condition is given making use of the theory of eigenvalues.

Let us give the theorem on the existence of a weak solution to the problem (2).

Theorem 1. *In the problem* (2) *assume that conditions* (1) *are satisfied and the embedding* $V \subset H$ *is compact. The function* $f \in L_2(0,T;H)$*, the function* $p(t)$ *is absolutely continuous and non-increasing and takes positive values on the interval* $[0,T]$*. Let* $\overline{u} \in V$*. Then problem* (2) *has a unique solution* $u(t)$ *such that* $u \in L_2(0,T;V) \cap C([0,T],H)$*,* $u' \in L_2(0,T;V')$*. Moreover, the following estimate holds*

$$\max_{0\leqslant t\leqslant T} \|u(t)\|_H^2 + \int\limits_0^T \Big(\|u(t)\|_V^2 + \|u'(t)\|_{V'}^2\Big)\,dt \leqslant C\Big(\|\overline{u}\|_V^2 + \int\limits_0^T \|f(t)\|_H^2\,dt\Big).$$

In following theorems we obtained better smoothness of the solution to (2) than in Theorem 1.

Theorem 2. *Let the assumptions of Theorem 1 hold. In the problem* (2) *assume that the function* $f(t)$ *belongs to the class* $L_1(0,T;H)\bigcap L_2(0,T;V')$*, the derivative* f' *belongs to* $L_2(0,T;V')$ *and* $Af \in L_1(0,T;H)$*. Let* $f(0)\in H$ *and* $\overline{u} \in D(A) = \{u \in V : Au \in H\}$*. Then the weak solution* $u(t)$ *to the problem* (2) *possesses the additional smoothness* $u' \in L_2(0,T;V)\bigcap C([0,T],H)$ *and* $u'' \in L_2(0,T;V')$*. Moreover, the following estimate holds*

$$\max_{0\leqslant t\leqslant T} \|u'(t)\|_H^2 + \int\limits_0^T (\|u'(t)\|_V^2 + \|u''(t)\|_{V'}^2)\,dt \leqslant K\Big\{\|A\overline{u}\|_H^2 + \Big(\int\limits_0^T \|Af(t)\|_H\,dt\Big)^2 + \\ + \int\limits_0^T (\|f(t)\|_{V'}^2 + \|f'(t)\|_{V'}^2)\,dt + \|f(0)\|_H^2\Big\}.$$

We call the form $a(u,v)$ Hermitian, if for all $u,v \in V$ $a(u,v) = \overline{a(v,u)}$, where the bar means complex conjugation.

Theorem 3. *Let the assumptions of Theorem 1 hold. Let the form* $a(u,v)$ *be Hermitian. In the problem* (2) *assume that the function* $f(t)$ *belongs to the class* $L_1(0,T;V)\bigcap L_2(0,T;H)$ *and the element* $\overline{u} \in V$*. Then the weak solution* $u(t)$ *to the problem* (2) *belongs to* $C([0,T],V)$ *and* u'*,* $Au \in L_2(0,T;H)$*. Moreover, the following estimate holds*

$$\max_{0\leqslant t\leqslant T} \|u(t)\|_V^2 + \int\limits_0^T \Big(\|u'(t)\|_H^2 + \|Au(t)\|_H^2\Big)\,dt \leqslant \\ K\Big\{\|\overline{u}\|_V^2 + \Big(\int\limits_0^T \|f(t)\|_V\,dt\Big)^2 + \int\limits_0^T \|f(t)\|_H^2\,dt\Big\}.$$

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DAMPING EFFECT OF CAPUTO FRACTIONAL TIME DERIVATIVES IN NONLINEAR WAVE EQUATIONS

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In the past few decades, there has been an extraordinary number of papers in the international literature on the application of fractional calculus to find new solutions to a great many differential equations with applications to practically every branch of science. There occurs, however, an important discrepancy between the applications of ordinary and fractional calculus in the case of conservative systems that possess families of periodic solutions: When fractional time derivatives of the Caputo type are employed, even though the system remains conservative, all solutions are seen to dissipate to zero or a nontrivial steady state, as soon as the order of the time derivative is less than two [1]. In this talk, I will explore this phenomenon further in the case of breather solutions of a Klein Gordon nonlinear wave equation [2].

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HYPERELLIPTIC SIGMA FUNCTIONS AND THE SEQUENCE OF THE NOVIKOV'S G-EQUATIONS

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In 1974, S. P. Novikov discovered an algebro-geometrical method for constructing periodic and quasi-periodic solutions of the KdV equation. He introduced the g-stationary equations of the KdV-hierarchy (namely the Novikov's g-equations) which correspond to integrable polynomial dynamical systems in $\mathbb{C}^{3g}$ with $2g$ polynomial integrals.

The talk is devoted to differential equations and dynamical systems, which are integrable in hyperelliptic sigma functions.

We will introduce systems of $2g$-dimensional heat equations in a nonholonomic frame which define this functions. The operators of such system generate a polynomial Lie algebra with only three generators for $g > 1$. We will construct an infinite-dimensional polynomial dynamical system that is universal for all polynomial dynamical systems corresponding to the sequence of Novikov's g-equations.

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POLYNOMIAL DYNAMICAL SYSTEMS CORRESPONDING TO DIFFERENTIATIONS OF HYPERELLIPTIC FUNCTIONS AND KORTEWEG–DE VRIES HIERARCHY

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We will describe the correspondence of the universal bundle of Jacobians of genus g hyperelliptic curves with the polynomial map

$$\rho_g : \mathbb{C}^{3g} \to \mathbb{C}^{2g}$$

that expresses the $2g$ parameters of the hyperelliptic curve of genus g as polynomials with rational coefficients in the generators of the field of hyperelliptic functions. We will give the relation of this map to the Korteweg–de Vries hierarchy following [1].

Our investigation of explicit formulas for differentiation of hyperelliptic functions, see [2], allowed us to obtain the polynomial dynamical systems in $\mathbb{C}^{3g}$ that correspond to differentiations of hyperelliptic functions, see [3]. We will show explicit formulas for the polynomial dynamical systems, their infinite-dimensional generalizations and the connection to Novikov equations.

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LAGRANGIAN MULTIFORMS: A VARIATIONAL FRAMEWORK FOR INTEGRABLE HIERARCHIES

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After a short review of the notion of Lagrangian multiforms as a variational framework for integrable hierarchies, I will present a general construction for finite dimensional integrable hierarchies. It relies on Semenov-Tian-Shanski's concept of Lie dialgebra. The classical Yang-Baxter equation underlies important properties of our Lagrangian multiforms. Thus, it is cast as a pillar of a Lagrangian framework, similar to its central role in the Hamiltonian framework. Examples of the construction will be given which provide Lagrangian multiforms for various famous hierarchies: open Toda chain, (cyclotomic) Gaudin model and periodic Toda chain. If time allows, I will briefly mention how the same underlying algebraic structures allow one to construct Lagrangian multiforms for integrable ultralocal field theories in $1 + 1$ dimensions.

This is based on joint works with Marta Dell'Atti, Anup Singh, Matteo Stoppato and Benoit Vicedo, [1–3].

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NONCOMMUTATIVE SOLUTIONS OF LOCAL TETRAHEDRON EQUATION

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This talk is related to n-simplex equations and local n-simplex equations. They are generalization of well-known Yang-Baxter and Zamolodchikov equations [3 – 5] (also Lax equation and local Yang–Baxter equation for local equations), which have applications in many fields of mathematics and physics, including statistical mechanics, quantum field theory and integrable systems.

In this talk we are especially interested in n-simplex maps on noncommutative division ring. Notable examples of division rings are a noncommutative group and a ring of endomorphisms of a prime module.

Let $\mathcal{X}$ be a set. Map $S : (x, y, z, t) \mapsto (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t), r(x, y, z, t))$ is called *4-simplex map* if it satifsfies *4-simplex equation*

$$S^{1234} \circ S^{1567} \circ S^{2589} \circ S^{368,10} \circ S^{479,10} = S^{479,10} \circ S^{368,10} \circ S^{2589} \circ S^{1567} \circ S^{1234}.$$

Now, let $\mathrm{L} = \mathrm{L}(x)$ be 3×3 matrix, which depends on variable $x \in \mathcal{X}$

$$\mathrm{L}(x) = \begin{pmatrix} a(x) & b(x) & c(x) \\ d(x) & e(x) & f(x) \\ k(x) & l(x) & m(x) \end{pmatrix}$$

. Let $\mathrm{L}^6_{ijk}(x)$, $i, j, k = 1, \dots 6$, $i < j < k$, — 6×6 matrix extensions of $\mathrm{L}(x)$.

We call this matrix equation

$$\mathrm{L}^6_{123}(u)\mathrm{L}^6_{145}(v)\mathrm{L}^6_{246}(w)\mathrm{L}^6_{356}(r) = \mathrm{L}^6_{356}(t)\mathrm{L}^6_{246}(z)\mathrm{L}^6_{145}(y)\mathrm{L}^6_{123}(x)$$

local tetrahedron equation. Solutions to this equation may generate solutions to 4-simplex equation.

We would have two main contexts — X may be a field or a noncommutative division ring. In this talk we will present which matrices of the type

$$\mathrm{L}(x_{11}, \ldots, x_{33}) = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

may lead to new solutions of 4-simplex equations i.e. they have solution to local tetrahedron equation if X is a division ring. We study difference in classification between commutative case which was studied in [1] and noncommutative case. Additionally, we introduce procedure to get novel 4-simplex maps associated with known tetrahedron maps from [2]. Also, we introduce conditional n-simplex maps and study it with examples of 4-simplex maps. Lastly, several innovative 4-simplex maps on noncommutative groups and division rings are constructed.

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CONSTRUCTIONS OF NEW NONLINEAR INTEGRABLE DIFFERENTIAL-DIFFERENCE EQUATIONS AND MIURA-TYPE TRANSFORMATIONS

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This talk is devoted to (nonlinear) integrable differential-difference equations and their Miura-type transformations (MTs). For a given equation E, equations obtained from E by applying MTs are called *modified equations corresponding to* E.

MTs possess the following remarkable properties:

- Applying MTs to a given integrable equation, one gets modified equations which are integrable as well.
- One can often obtain an auto-Bäcklund transformation for a given equation E by using two MTs for E.

In this talk I will demonstrate methods for constructing new nonlinear integrable equations connected by new MTs to known equations.

DEGENERATE BILLIARDS WITH SEMI-RIGID WALLS AND NONLINEAR SHORE WAVES

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By billiards with semi-rigid walls we mean special solutions of Hamiltonian systems defined on a two-dimensional plane (x_1, x_2) by Hamiltonians $H = D(x_1, x_2)(p_1^2+p_2^2)$, where $D(x)$ is a smooth function that vanishes on some smooth closed curve Γ, with $\nabla D\Big|_\Gamma \neq 0$. Such billiards arise in the theory of waves on water in limited and unlimited basins, the function D describes the bottom of the basin and inside the basin takes positive values. The curve Γ is the coastline. In the work [1], under the condition of integrability of the Hamiltonian system with the Hamiltonian H, using the semiclassical approximation and the modified Carrier-Greenspan transformation, time-periodic asymptotic solutions of a nonlinear system of shallow water equations in basins with shallow shores localized in the vicinity of the coastline were constructed. The corresponding trajectories of the Hamiltonian system form non-compact ("non-standard") Liouville tori, their projections on the plane (x_1, x_2) sweep the annular area and reflect off some simple caustics located inside the pool and the shoreline, which is a "non-standard" caustic. The defect of the solutions constructed in [1] consists in the requirement of integrability of the introduced Hamiltonian system, which practically cannot be fulfilled in real situations. In this talk, we mainly consider degenerate situations [2] when "standard" caustics are very close to the coastline ("non-standard" caustics). Then ‘fast and slow variables" appear in the Hamiltonian system, the requirement of integrability then disappears, and it is always possible to construct effective asymptotic wave solutions with a small number of oscillations normal to the shore (which are analogs of Stokes and Ursell waves). The corresponding trajectories are strongly localized in the narrow vicinity of the coast, while they always enter the coastline and reflect from it at an angle of 90 degrees. Thus, we have asymptotic solutions similar to the "whispering gallery" type solutions known in acoustics, but at the same time, their existence due to the "degenerate" wall (coastline) does not require the convexity of the two-dimensional region in which the pool is located, that is, the region on the two-dimensional plane (x_1, x_2), in of which $D(x_1, x_2) > 0$. The examples show the dependence of the local amplitude of nonlinear waves on the angle of inclination, curvature of the coastline, etc.

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MULTIDIMENSIONAL RIEMANNIAN METRICS FOR INTEGRATING THE NAVIER–STOKES EQUATIONS

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The properties of $14D$-metric which is the Ricci-flat $R_{ik} = 0$ on solutions of the Navier-Stokes equations in the Euler variables are studied.

For the study properties of Navier-Stokes equations in Lagrangian variables the $6D$-metric is introduced and examples of their particular solutions are constructed.

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NUMERICAL STUDY OF STABLE REGIMES AND BIFURCATIONS IN A DYNAMICALLY SELF-ORGANIZING SYSTEM

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The system under consideration is described by the following equations:

$$\frac{du_n}{dt} = -N(v_{n+1} - v_n) + \mu u_n - u_n^3, \quad \frac{dv_n}{dt} = -N(u_n - u_{n-1}), \tag{1}$$

$$u_0 = 0, \quad v_{N+1} = \beta \frac{du_N}{dt}, \quad n = 1, 2, \ldots, N. \tag{2}$$

This model describes the behavior of an autogenerator, which consists of a chain of N sequentially coupled simplified FitzHugh-Nagumo neurons. A resistor and a constant power source are connected to one end of the chain, while a fixed capacitor is connected to the other end.

Of particular interest is the regime where $N \gg 1$, prompting the transition to a continuous model:

$$\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x} + \mu u - u^3, \quad \frac{\partial v}{\partial t} = -\frac{\partial u}{\partial x}, \tag{3}$$

$$u|_{x=0} = 0, \quad \left(\beta \frac{\partial v}{\partial x} + v\right)\Bigg|_{x=1} = \beta(\mu u - u^3)|_{x=1}. \tag{4}$$

In the continuous model, the quasinormal form is derived at the critical values of the parameter using standard methods.(see, for example, [1], [2]):

$$\dot{\eta}_l = \left[\delta_l - d_l\, \eta_l - \sum_{\substack{m=1 \\ m \neq l}}^{\infty} d_{l,m}\, \eta_m\right] \eta_l, \quad l \geq 1. \tag{5}$$

In the first stage of the study, conditions for the existence and stability of equilibrium states are determined for the quasinormal form.

Numerical analysis has shown that only single-mode equilibrium states are stable.

The next stage includes a large-scale numerical experiment to study the bifurcations of periodic single-mode regimes, which were constructed using analytical formulas.

It has been shown that for system (1)-(2), stable invariant tori and cycles are present at various values of the parameter μ.

The results of the study suggest that all oscillators included in the network are not generators (the only stable state is the zero equilibrium), but when coupled in the network, they exhibit complex behavior.

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NONLINEAR INTERACTIONS OF SOLITONS AND EXTERNAL FORCES

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We consider the interactions between solitons and external forces in various nonlinear equations, including the Benjamin-Ono (BO) equation, the Schamel equation, and the modified Korteweg-de Vries (mKdV) equation. Assuming a weak external force, we derive a dynamical system that governs the evolution of the soliton amplitude and position. The dynamical system predicts (i) resonance between the soliton and the external force, (ii) oscillatory motion characterized by closed orbits, and (iii) displacement from the initial position while maintaining the soliton direction. However, numerical simulations reveal the emergence of an unstable spiral pattern instead of closed orbits. Furthermore, we also consider the external force to be random. In particular, for the BO equation, we demonstrate that randomness primarily manifests in the soliton phase. Assuming a uniform distribution for the soliton phase, we analytically compute the averaged soliton field and its statistical moments. Under these conditions, we show that the averaged soliton field spreads and dampens.

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PERIODIC REGIMES OF MULTICLUSTER SYNCHRONIZATION IN COUPLED NETWORKS OF NONLINEAR OSCILLATORS WITH INTEGRAL

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A fully coupled network of nonlinear oscillators of the form

$$\dot{x}_j = F_j(x_j, u_j), \quad j = 1, 2, \dots, m \tag{1}$$

is considered. Here, $m \geq 2$, $x_j = x_j(t) \in \mathbb{R}^n$, $n \geq 2$, the dot over x_j is used to denote differentiation with respect to t, $u_j = \sum\limits_{s=1\ s\neq j}^{m} G_s(x_s)$, and the vector functions $F_j(x, u)$, $G_j(x)$, $j = 1, 2, \dots, m$, with values in $\mathbb{R}^n$, are infinitely differentiable by their variables $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

Let the partial systems, which correspond to (1)

$$\dot{x} = F_j(x, 0), \quad j = 1, 2, \dots, m \tag{2}$$

have an exponentially orbitally stable cycle. We consider the situation, when oscillators (2) interact with each other according to the principle “everyone with everyone”. This problem was introduced by the authors in the articles [1–3]. The conditions for the existence and stability of traveling waves and two-cluster synchronization are found.

In this work we consider a generalization of the problem (1). We now consider the case of a continuum number of oscillators in the network, and arrive at an integro-differential equation of the form

$$\frac{\partial x}{\partial t} = F(x, u), \tag{3}$$

where $u = \int_0^1 G(x(t, s))ds$, and, for every $t > 0$, $x(t, s) \in L_\infty\big([0, 1]; \mathbb{R}^n\big)$, $n \geq 2$.

Let us introduce the definition of the multi-cluster synchronization mode.

Definition. *Let* $[0, 1] = \bigcup\limits_{k=1}^{r} A_k$*, where* A_k *are Lebesgue measurable with positive measure. A periodic solution* $x_0(t, s)$ *of Eq.*(3) *is called a multicluster synchronization mode, if the following is true:*

$$x_0(t, s) = \{V_k(t) \text{ for } s \in A_k,\ k = 1, \ldots, r\},$$

almost everywhere with respect to s*,* $V_k(t) \equiv V_k(t + T)$*, where* T *is a period,* $V_k(t) \neq V_j(t)$ *for every* $t > 0$*,* $k \neq j$*.*

In this work, conditions for the existence and stability of a continuous family of multi-cluster synchronization modes are found.

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ONE METHOD FOR THE VERIFICATION OF HYPERBOLICITY

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We consider an arbitrarily chosen open set $U \subset M$, where M is a smooth Riemannian manifold of dimension m, $m \geq 2$, and a map $f : U \to M$ from C^1, that is a diffeomorphism from U to $f(U) \subset M$. Assume that there exists a compact subset $A \subset U$, such that $f(A) = A$. In this work we present two sets of sufficient conditions under which the invariant set A of the diffeomorphism f has the hyperbolicity property. Along with that, a new method is developed for the verification of hyperbolicity, alternative to the well-known cone criterion (see [1]).

Everywhere below, by $||\cdot||$ we denote the Riemannian norm in the tangent space T_xM, induced by some Riemannian metric on M (we omit the dependence of this norm on $x \in M$ for brevity). For each point $x \in A$ we define the operators

$$D(f^n(x)) = Df(x_{n-1}) \circ Df(x_{n-2}) \circ \ldots \circ Df(x_0),$$
$$D(f^{-n}(x)) = [Df(x_{-n})]^{-1} \circ [Df(x_{-(n-1)})]^{-1} \circ \ldots \circ [Df(x_{-1})]^{-1},\ n \in \mathbb{N},$$

where $Df(x) : T_xM \to T_{f(x)}M$ is the differential of the map f, $x_j = f^j(x)$, $j \in \mathbb{Z}$.

Definition (U). An invariant set A is called *hyperbolic* for the map f, if, firstly, for every $x \in A$ the tangent space T_xM can be represented as a direct sum $T_xM = E^u_x \oplus E^s_x$ of linear subspaces E^u_x, E^s_x, with the invariance properties

$$Df(x)E^u_x = E^u_{f(x)}, \quad Df(x)E^s_x = E^s_{f(x)} \quad \forall\, x \in A;$$

secondly, there exist constants $\mu_1, \mu_2 \in (0,1)$, $c_1, c_2 > 0$, such that

$$||D(f^{-n}(x))\xi|| \leq c_1\mu_1^n||\xi|| \quad \forall\, x \in A, \quad \forall\, \xi \in E^u_x, \quad \forall\, n \in \mathbb{N},$$

$$||D(f^{n}(x))\xi|| \leq c_2\mu_2^n||\xi|| \quad \forall\, x \in A, \quad \forall\, \xi \in E^s_x, \quad \forall\, n \in \mathbb{N}.$$

Recall that in [2] hyperbolic diffeomorphisms were referred to as *U - diffeomorphisms* or *U-systems.*

We now proceed to describing the constraints which ensure the hyperbolicity of the invariant set A.

Condition 1 *For every $x \in A$, the decomposition holds*

$$T_xM = E_1(x) \oplus E_2(x), \tag{1}$$

where $\oplus$ is direct sum of the linear subspaces $E_1(x)$, $E_2(x)$ which, generally speaking, are not Df-invariant and not necessarily continuous on $x \in A$. Their dependence on x is such that for the corresponding projectors

$$\begin{gathered} P(x),\ Q(x):\ \ \forall\,\xi = \xi_1(x) + \xi_2(x), \quad \xi_1(x) \in E_1(x), \quad \xi_2(x) \in E_2(x), \\ P(x)\xi = \xi_1(x), \quad Q(x)\xi = \xi_2(x) \end{gathered} \tag{2}$$

the following inequalities hold

$$\sup_{x\in A} ||P(x)||_{T_xM\to T_xM} < \infty, \quad \sup_{x\in A} ||Q(x)||_{T_xM\to T_xM} < \infty. \tag{3}$$

Since the projectors $P(x)$ and $Q(x)$ are not continuous, conditions (3) are not automatically satisfied. Therefore, we require their fulfillment. The geometric meaning of these conditions consists in the separability from zero of the angle between the subspaces $E_1(x)$ and $E_2(x)$ in (1).

Using the decomposition (1) and projectors (2), let us introduce the operators

$$\Lambda_{j,1}(x) = P(f(x))Df(x):\ E_j(x) \to E_1(f(x)), \quad j = 1, 2, \tag{4}$$

$$\Lambda_{j,2}(x) = Q(f(x))Df(x):\ E_j(x) \to E_2(f(x)), \quad j = 1, 2, \tag{5}$$

Condition 2 *Assume that for every $x \in A$ the inequality* $\dim E_1(x) > 0$ *holds, the linear operator $\Lambda_{1,1}(x)$ from* (4) *is invertible, and*

$$\sup_{x\in A} ||\Lambda^{-1}_{1,1}(x)||_{E_1(f(x))\to E_1(x)} < \infty. \tag{6}$$

Formulas (4) – (5) and condition 2 allow to define the sets of constants

$$\alpha_1 = \sup_{x\in A} ||\Lambda^{-1}_{1,1}(x)||_{E_1(f(x))\to E_1(x)},\ \alpha_2 = \sup_{x\in A} ||\Lambda_{2,2}(x)||_{E_2(x)\to E_2(f(x))}, \tag{7}$$

$$\beta_1 = \sup_{x\in A} ||\Lambda_{1,2}(x)||_{E_1(x)\to E_2(f(x))},\ \beta_2 = \sup_{x\in A} ||\Lambda_{1,1}^{-1}(x)\Lambda_{2,1}(x)||_{E_2(x)\to E_1(x)}, \tag{8}$$

$$\gamma_1 = \sup_{x\in A} ||\Lambda_{1,2}(x)\Lambda_{1,1}^{-1}(x)||_{E_1(f(x))\to E_2(f(x))},\ \gamma_2 = \sup_{x\in A} ||\Lambda_{2,1}(x)||_{E_2(x)\to E_1(f(x))}, \tag{9}$$

The following statements holds.

Theorem 1 *Let the conditions 1, 2 and the inequalities*

$$\alpha_1 < 1, \quad \alpha_2 < 1, \quad \min(\beta_1\beta_2, \gamma_1\gamma_2) < (1-\alpha_1)(1-\alpha_2), \tag{10}$$

hold, where α_j, β_j, γ_j, $j = 1,2$ *are given by* (7) – (9). *Then, the invariant set* A *of the diffeomorphism* f *is hyperbolic.*

Similar results to Theorem 1 are given in [3,4], where, using the singularities of the area of the ring for diffeomorphisms, sufficient conditions for the hyperbolicity of the attractor are obtained.

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HYPERBOLIC STABLE POLYNOMIALS AND TOTAL POSITIVITY

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In my report I will express the condition for hyperbolicity and stability of a real polynomial in terms of the oscillation of some matrix associated with it. The result uses the Sturm chain technique as well as the classical theorems and statements about totally positive matrices formulated by Gantmacher and Krein. The resulting condition is closely related to Toeplitz positivity and the problem of parametrizing special cells of positive manifolds.

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ON LYAPUNOV DIMENSION OF DIFFUSION CHAOS IN ONE ECOLOGICAL MODEL

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This work is about a multimode diffusion chaos at low values of diffusion parameter in a spatially distributed Hutchinson equation, which describes the population density dynamics on a line segment:

$$\frac{\partial N}{\partial t} = D\frac{\partial^2 N}{\partial x^2} + r\,(1 - N_{t-1})\,N, \qquad \left.\frac{\partial N}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial N}{\partial x}\right|_{x=1} = 0.$$

Here $N(t,x)$ is the population density at time t and point x of a line segment Ω ($\Omega = \{x \mid 0 \le x \le 1\}$), D is a diffusion coefficient, r is a Malthusian coefficient of linear growth, $N_{t-1} \equiv N(t-1,x)$. Some analytical and numerical results are described in [1, 2] in more complex spatially case.

If we consider differential-difference model, we can calculate the values close to Lyapunov exponents and thus evaluate Lyapunov dimension (see [3, 4]), using the algorithm that is described in paper [5].

Graphs of the dependence of the approximate Lyapunov dimension on the diffusion coefficient and on the number of partition points were constructed, that show the existence of multimode diffusion chaos when diffusion parameter D is close to zero.

In addition, it is possible to prove the closeness of the dynamical properties of the continuous model and its difference approximation.

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BIFURCATIONS AND CHAOTIC ATTRACTORS OF FOUR DIMENSIONAL HENON-LIKE MAP WITH POLYNIMIAL NONLINEARITY

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The generalized Henon map has the form

$$T: \quad \begin{array}{rcl} \bar{x} = & f(x) + 1y & \triangleq \; g(x,y), \\ \bar{y} = & \mathbf{b}x + Ay & \triangleq \; L(x,y), \end{array} \tag{1}$$

where $x \in R^1$, $y = \text{column}(y_1, y_2, \dots, y_n) \in R^n$, 1 is the all ones line of length n, $f(x)$ is a continuous smooth function, $\mathbf{b} = \text{column}(\mu, 0, 0, \dots, 0)$, μ is the parameter, A is contructing matrix having zero first row.
In this talk we consuder a particular four dimentional case of this map then $f(x)$ is a polynomial function and μ is a small parameter.

Using the small parameter method [1,2], we derive bifurcations and chaotic attractors of this map reconstructed from those of one-dimensional map $x \to f(x)$.

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REDUCTION OF THE LAPLACE SEQUENCE AND SINE-GORDON TYPE EQUATIONS

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It is well known that the cascade method of Laplace integration is an effective tool for constructing solutions to linear equations of hyperbolic type, as well as nonlinear equations of Liouville type. The connection between Laplace's method and soliton equations of hyperbolic type remains less studied. In a series of our works ([1]-[3]), it was shown that the sequence of Laplace transforms also has important applications in the theory of hyperbolic equations of soliton type. Namely, that the sequence of Laplace provides a simple way to construct such fundamental objects related to the theory of integrability as the recursion operator, the Lax pair and equations of Dubrovin type, allowing one to find algebraic-geometric solutions. As a result of applying this approach, previously unknown recursion operators and Lax pairs were found for several nonlinear integrable equations of the sine-Gordon type.

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NONCOMMUTATIVE $N-$TORUS IN A MAGNETIC FIELD: VOLUME INVARIANCE, SCALAR CURVATURE AND QUANTUM STOCHASTIC EQUATION

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Motivated by the works of Chakraborty *et al* [*J. operator theory,* **49**(2003), 185 − 201] and Sakamoto and Tanimura [*J. Math. Phys.,* **44**, (2003), 5042], we investigate the noncommutative $n-$ torus in a magnetic field. We study the invariance of volume, integrated scalar curvature and volume form using the method of perturbation by inner derivation of the magnetic Laplacian in this geometric framework. Moreover, we derive the magnetic stochastic process describing the motion of a particle in a uniform magnetic field in this torus and deduce the properties of the solution of the corresponding magnetic quantum stochastic differential equation.

DISCRETE MIURA-TYPE TRANSFORMATIONS, GAUGE SIMPLIFICATIONS, AND GROUP ACTIONS ASSOCIATED WITH LAX REPRESENTATIONS FOR DIFFERENTIAL-DIFFERENCE EQUATIONS

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In this talk I will present new results on relations between differential-difference matrix Lax representations, gauge transformations, and discrete Miura-type transformations, which belong to the main tools in the theory of (nonlinear) integrable differential-difference equations. Such equations occupy a prominent place in the modern theory of integrable systems and are presently the subject of intensive study. In particular, such equations arise as discretizations of integrable PDEs and various geometric constructions and as chains associated with Darboux transformations of PDEs (see, e.g., [1, 2, 3, 4] and references therein).

I will present sufficient conditions for the possibility to simplify a differential-difference matrix Lax representation by local matrix gauge transformations. Also, I will present a method to construct Miura-type transformations for differential-difference equations, using gauge transformations and invariants of Lie group actions on manifolds associated with Lax representations of such equations.

The method is applicable to a wide class of Lax representations. The considered examples include the (modified) Volterra, Itoh-Narita-Bogoyavlensky, Belov-Chaltikian, Toda lattice equations and Adler-Postnikov equations from [5] as well as the equation (introduced by G. Marí Beffa and Jing Ping Wang [3]) which describes the evolution induced on invariants by an invariant evolution of planar polygons. Applying our method to these examples, one obtains new integrable nonlinear differential-difference equations connected with these equations by new Miura-type transformations.

Some steps of our method generalize (in the differential-difference setting) a result of V. G. Drinfeld and V. V. Sokolov [6] on Miura-type transformations for the partial differential Korteweg–De Vries equation.

This talk is based on the paper [7] and new developments of results from a joint work with G. Berkeley [8].

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DYNAMICS OF ONE CAUCHY PROBLEM WITH IMPULSES

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Let us consider a fully connected association of singularly perturbed differential equations with a delay presented as a mathematical model of the impulse system [1]

$$\dot{u}_j = d\sum_{\substack{s=1\\ s\neq j}}^{m} \sigma\left(\frac{u_s}{u_j}\right) + \lambda(-1+\alpha f(u_j(t-1)) - \beta g(u_j))u_j, \quad j = \overline{1,m}, \quad (1)$$

where $u_j = u_j(t) > 0$, $m = 3$, real parameters $d > 0$, $\lambda \gg 1$, $\beta > 0$, $\alpha > 1 + \beta$, coupling function $\sigma(u)$ is presented as

$$\sigma(u) = \frac{\delta(u-1)}{u+\delta}, \qquad \delta \in (0, 2-\sqrt{3}) \cup (2+\sqrt{3}, +\infty)$$

and smooth functions $f(u), g(u) \in C^2(\mathbb{R}_+)$ have the following conditions

$$0 < \beta g(u) < \alpha, \quad f(0) = g(0) = 1, \quad \forall u \in \mathbb{R}_+,$$
$$f(u), g(u), uf'(u), ug'(u), u^2 f''(u), u^2 g''(u) = O(1/u) \text{ for } u \to +\infty.$$

For system (1) there were researched tasks of existence, stability and asymptotic representation of periodic solutions based on a bifurcation analysis of the special two-dimensional map [1]

$$\Phi(z) : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} y_1(T_0, z_1, z_2) \\ y_2(T_0, z_1, z_2) \end{pmatrix},$$

where $T_0 = \alpha + 1 + (\beta+1)/(\alpha - \beta - 1)$ is the first approximation of a stable cycle of a single oscillator in system (1) and functions $y_1(t, z_1, z_2)$, $y_2(t, z_1, z_2)$ have entry conditions $y_1(-0) = z_1, y_2(-0) = z_2$. The main focus of this research was on the number of coexisting stable regimes. Some results of this research for the certain values of parameters α, β, δ were published in article [2].

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ASYMPTOTIC INTEGRABILITY

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In this talk, we shall consider a property of nonlinear wave equations called their *asymptotic integrability*. It means that two asymptotic limits of the wave motion, namely, the dispersionless evolution of smooth pulses and the propagation of high-frequency wave packets, are integrable in the sense that the Hamilton equations for a wave packet have an integral of motion for any dispersionless evolution of the background flow [1,2]. It turns out that in case of equations with two wave variables such an integral does only exist for very special forms of the dispersion relation for linear waves. It is shown [3] that the expression for this integral is related with the quasi-classical limit of the Lax spectral problem in the Ablowitz-Kaup-Newell-Segur scheme. The quasi-classical limit of the second equation of the Lax pair is equivalent to the 'number of waves' conservation law. This approach allows one to obtain dispersive generalizations of hydrodynamic equations. The theory is also illustrated by applications to the Hamiltonian theory of propagation of solitons along non-uniform and time-dependent background wave [4,5].

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NETWORK MODEL OF LIMBIC SEIZURE PROPAGATION

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Limbic epilepsy also known as temporal lobe epilepsy is the most widespread and clinically relevant epilepsy form [1]. Though the clinical study of limbic seizures has been long and persistent [2], there is no mathematical models of seizure generation.

Here, we propose a new approach to mathematical modeling the limbic seizures. As the first step we constructed a central pattern ring generator to provide a main frequency [1]. This generator consists of a number (usually from six to some dozens) of hippocampal pyramid neurons. For all neurons the models in the Hodgkin–Huxley formalism were written down and the synapses were established in the form of sigmoid function and delay line, given a system of DDEs for the whole ensemble. The analytical hypothesis was established for the dependency of the main frequency on the number of neurons and the delayed in the coupling. This hypothesis was tested for a number of regimes using calculated series and was shown to well (with a relative square error about 10^{-4}) match the simulations.

Then, we made a next step and provided a model of synchronization of surrounding areas (an ensemble of hippocampal and cortical mathematical neurons) by the central pattern generator. The ability of the ensemble to be synchronized significantly depends on its network architecture (directed graph, or matrix of connections). To study which connectivity matrices correspond to better synchronization and therefore may be considered as models of epileptic brain structures, we studied spectra of eigenvalues of all randomly generated connectivity matrices as well as of matrices constructed from those be adding of deleting some connections (small perturbations). As a result we propose a relatively small ($\sim 10^2$ oscillators) network dynamical model of limbic seizure propagation constructed in the form of DDEs base on anatomical rules and using specific biophysical models for each neuron type.

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TRAVELLING WAVES IN THE RING OF COUPLED OSCILLATORS WITH DELAYED FEEDBACK

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We study the travelling waves in the unidirectionally coupled ring of N oscillators with delay of the form

$$\dot{x}_k + x_k = \lambda F(x_k(t - T_1)) + \gamma(x_{k-1} - x_k), \quad k = 1, \dots, N, \quad x_0 \equiv x_N,$$

where parameter λ is positive and sufficiently large ($\lambda \gg 1$), delay time T_1 and coupling parameter γ are positive, and feedback function F is a compactly supported positive function, that is

$$F(y) = \begin{cases} f(y), & \text{if } y \in [-p, p], \\ 0, & \text{if } y < -p \text{ or } y > p, \end{cases}$$

where p is some positive constant and $f(y) > 0$ for all $y \in (0, p]$, $f(y)$ is a piece-wise continuous and bounded function on the segment $y \in [-p, p]$.

This problem was reduced to studying dynamics of equation with two delays

$$\dot{x} + x = \lambda F(x(t - T_1)) + \gamma(x(t - T_2) - x).$$

Using special asymptotic method of large parameter we prove that this equation has a relaxation cycle and study its properties: amplitude, period, asymptotics. The sufficient conditions of stability are found. Based on this periodic solution the travelling waves of initial model were constructed.

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DYNAMICS OF FULL-COUPLED CHAINS OF A GREAT NUMBER OF OSCILLATORS

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The subject of this work is the study of local dynamics of full-coupled chains of a great number of oscillators with a large delay in couplings. From a discrete model describing the dynamics of a great number of coupled oscillators, a transition has been made to a nonlinear integro-differential equation, continuously depending on time and space variable. A class of full-coupled systems has been considered. The main assumption is that the amount of delay in the couplings is large enough. This assumption opens the way to the use of special asymptotic methods of study. The parameters under which the critical case is realized in the problem of the equilibrium state stability have been distinguished. It is shown that they have infinite dimension. The analogues of normal forms – nonlinear boundary value problems of Ginzburg–Landau type have been constructed. In some cases, these boundary value problems contain integral components too. Their nonlocal dynamics describes the behavior of all solutions of the original equations in the balance state neighbourhood.

As applied to the considered problems, methods of constructing quasinormal forms on central manifolds are developed. An algorithm for constructing the asymptotics of solutions based on the use of quasinormal forms for determining slowly varying amplitudes has been created.

Quasinormal forms that determine the dynamics of the original boundary value problem have been constructed. The dominant terms of asymptotic approximations for solutions of the considered chains have been obtained. On the basis of the given statements, a number of interesting dynamical effects have been revealed. For example, an infinite alternation of direct and reverse bifurcations when the delay coefficient increases. Their distinguishing feature is that they have two or three spatial variables.

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STABLE SOLUTIONS OF A NONLINEAR SYSTEM OF TWO EQUATIONS WITH DELAY

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Consider two coupled oscillators with delayed feedback

$$\begin{cases} \dot{u}_1 + u_1 = \lambda F(u_1(t-T)) + \gamma(u_2 - u_1), \\ \dot{u}_2 + u_2 = \lambda F(u_2(t-T)) + \gamma(u_1 - u_2), \end{cases} \tag{1}$$

where functions u_1 and u_2 are real and scalar.

Function $F(x)$ is defined by the formula

$$F(x) = \begin{cases} b, \ x < p_L, \\ f(x), \ x \in [p_L, p_R], \quad p_L < 0 < p_R, \\ d, \ x > p_R, \end{cases}$$

$$\lambda \gg 1, \quad \gamma \in (-\frac{1}{2}, 0) \cup (0, +\infty).$$

Function $f(x)$ — is nonlinear, piece-wise continuous, bounded and $f(x) \neq 0$ on any segment of non-zero length.

The conditions for the existence of the following five types of solutions were found and the asymptotics were constructed:
1) solutions tending to a positive constant λd;
2) solutions tending to a negative constant λb;
3) solutions whose components tend to constants of different signs;
4) homogeneous cycle;
5) unhomogeneous cycle.

For each type of solution, stability was investigated, including analytically finding the regions of attraction for constant solutions.

ON QUADRIRATIONAL PENTAGON MAPS

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We classify rational solutions of a specific type of the set theoretical version of the pentagon equation. That is, we find all quadrirational maps $R : (x, y) \to (u(x, y), v(x, y))$, where u, v are two rational functions on two arguments, that serve as solutions of the pentagon equation. Furthermore, provided a pentagon map that admits a partial inverse, we obtain genuine entwining pentagon set theoretical solutions. Finally, we show how to obtain Yang–Baxter maps from entwining pentagon map.

ELECTRICAL NETWORKS AND THE TOTALLY POSITIVE SYMPLECTIC MATRICES

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A square matrix is totally positive if all its minors are positive. The study of this type of matrices was motivated by some remarkable physical problems [3] and largely gave rise of the theory of totally positive Grassmannians [6]. One of the cornerstone result in the theory of totally positive matrices is the Loewner-Whitney theorem [6]:

Theorem 1. *Each totally positive matrix can be decomposed in a product of elementary Jacobi matrices and a diagonal matrix with positive parameters.*

It is also natural and interesting to clarify Theorem 1 for different subsemigroups of the semigroup of totally positive matrices. In the focus of my talk will be an attempt to do it for totally positive symplectic matrices, which we will be studied with circular electrical networks and their embeddings to totally positive Grassmannians [1], [4], [5]. Particularly, we will demonstrate that the problem of parameters computing mentioned in Theorem 1 in case of totally positive symplectic matrices can be considered as the well-known and very important black-box problem [2], which consists in recovering of conductivities of an electric network by its response matrix [2].

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TOPOLOGICAL MODELING OF SINGULARITIES WITH SADDLE AND FOCAL COMPONENTS VIA BILLIARDS

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Nondegenerate singularities of integrable Hamiltonian systems can be described in topological sense (up to fiberwise homeomorphisms) as direct or semi-direct products of a "more simple" singularities of systems of 1 d.o.f. (center or saddle components) or systems of 2 d.o.f. (focal components) and may be a regular foliation $D^k \times T^k$. In a semi-direct product case, a finite group acts on the product s.th. the action is free, preserves the rank of momentum map. This result for singularities of a general case in the semilocal sense (invariant neighbourhood of a singular fiber) was obtained by N.T.Zung [1]. For more details about topological approach to integrable Hamiltonian systems see [2].

In our talk we will discuss the realization problem of such singularities via billiards. For arbitrary nondegenerate semi-local singularities of systems with 3 degrees of freedom with saddle and focal components billiard systems with a potential field were constructed s.th. their Liouville foliation contains singularity fiberwise to the one. Such billiard system is determined on a flat confocal billiard domain or a locally-flat 3-dimensional CW-complex with permutations glued from such flat confocal domains by their common boundary 2-cells. Such class of systems considered by us is a multi-dimensional generalization of "billiard books" suggested earlier by V.Vedyushkina. Integrable billiard systems on such domains model a wide class of integrable Hamiltonian systems with 2 d.o.f. and their singularities.

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DYNAMICS OF NEURAL NETWORKS WITH ADAPTIVE DELAYS

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Neuronal plasticity allows brain to learn and adapt by remodeling of its structure. Apart from the well-known synaptic plasticity, there are many other types of plasticity whose nontrivial interplay is crucial for the implementation of various brain functions. One of the least studied types of plasticity is the activity-dependent myelination of axons [1] which makes synaptic delays adaptive. The present talk is devoted to the possible role of adaptive delay in the collective dynamics of neural networks. In contrast to the known results on stabilization of neural activity [2], we show that the delay adaptation may lead to the emergence of slow self-sustained oscillations [3].

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NEGATIVE FLOWS

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For an evolutionary system of equations possessing a recursion operator we can introduce the notion of negative symmetry. Negative symmetries are of interest because they are often already known important equations and additionally they are derivative functions for higher symmetries. And the commutativity of negative symmetries corresponds to the so-called 3D compatible continuous equations. Thus we study the negative symmetries for the KDV equation and for the sine Gordon equation. We obtain a convenient notation for reductions from an additional subalgebra. And we also find new examples of 3D compatible equations.

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SELF-SIMILARITY AND COMBINATORIAL CORRELATIONS IN THREE-DIMENSIONAL STATISTICAL PHYSICS MODELS

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The report concerns the newly discovered self-similarity spin transform on three-dimensional cubic lattices. In particular, it makes possible calculation of nontrivial spin correlations in a “combinatorial” model, in which all permitted spin configurations have equal probabilities.

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FAMILIES OF STEP SOLUTIONS OF QUASINORMAL FORMS FOR A SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS

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Consider the system of equations

$$\frac{\partial u}{\partial t} = (A_0+\varepsilon A_1)u+F_2(u,u)+F_3(u,u,u)+(D_0+\varepsilon D_1)\frac{1}{2\pi}\int\limits_0^{2\pi} u(t,x+s)dx,$$

where $u = u(t,x) \in \mathbb{R}^n$, A_0, A_1, D_0, D_1 are $n \times n$ matrices, $F_2(*,*)$, $F_3(*,*,*)$ are linear functions of their arguments, ε is small real number.

This system is considered with the periodic boundary condition

$$u(t,x+2\pi) \equiv u(t,x).$$

The zero solution of this boundary value problem is asymptotic stable, when eigenvalues of matrices

$$A_0 + \varepsilon A_1 + \frac{1}{2\pi}(D_0 + \varepsilon D_1)\int\limits_0^{2\pi} \exp(iks)ds \quad (k = 0, \pm 1, \pm 2, \dots)$$

have negative real part.

Note that if $k \neq 0$ then these matrices equal to $A_0 + \varepsilon A_1$, and if $k = 0$ then we have matrix $A_0 + \varepsilon A_1 + D_0 + \varepsilon D_1$. Thus in main part stability of the zero solution of boundary value problem depends on eigenvalues of matrices A_0 and $A_0 + D_0$.

Let all of the eigenvalues of matrix $A_0 + D_0$ have negative real part, matrix A_0 has one zero eigenvalue and other eigenvalues with negative real part. Then dynamics of solutions of boundary value problem near zero solution describes by one-dimensional boundary value problem

$$\frac{\partial \xi}{\partial \tau} = \xi - (\xi^3 - M(\xi^3)),\ \xi(\tau, x + 2\pi) \equiv \xi(\tau, x),\ M(\xi) = 0,$$

which called as quasinormal form. Here $\tau = \varepsilon t$, $\xi = \xi(\tau, x)$, $M(\xi) = \frac{1}{2\pi}\int\limits_0^{2\pi} \xi(\tau, x)dx$. Boundary value problems of this type have the solutions in the form of step functions. We can describe the dynamics of quasinormal form in the terms of α-stability of step solutions (see [1], [2]).

Theorem 1. *Quasinormal form has the family of step solutions*

$$\xi(\tau, x) = \begin{cases} \pm\frac{2\pi-\alpha}{\sqrt{4\pi^2-6\pi\alpha+3\alpha^2}}, & x \in [0, \alpha), \\ \mp\frac{\alpha}{\sqrt{4\pi^2-6\pi\alpha+3\alpha^2}}, & x \in [\alpha, 2\pi). \end{cases} \quad (1)$$

Solution of that family is asymptotic α-stable, if $\frac{2\pi}{3} < \alpha < \frac{4\pi}{3}$.

Now let matrix A_0 has a one simple pair of imaginary eigenvalues, other eigenvalues have negative real part. Then dynamics of boundary value problem describes by quasinormal form

$$\frac{\partial \xi}{\partial \tau} = \lambda\xi + \sigma(\xi|\xi|^2 - M(\xi|\xi|^2)) + \beta\xi M(|\xi|^2) + \gamma\bar{\xi}M(\xi^2),$$

$$\xi(\tau, x + 2\pi) \equiv \xi(\tau, x),\ M(\xi) = 0,$$

where $\operatorname{Re}\lambda > 0$, $\operatorname{Re}\sigma < 0$, $\operatorname{Re}\beta < 0$.

Theorem 2. *Quasinormal form has the family of solutions*

$$\xi(\tau, x) = \rho(x)e^{i\omega(\alpha)\tau},$$

where

$$\rho(x) = \begin{cases} \rho_1(\alpha), & x \in [0, \alpha), \\ \rho_2(\alpha), & x \in [\alpha, 2\pi). \end{cases}$$

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INVESTIGATION OF SELF-OSCILLATING SOLUTIONS OF A MATHEMATICAL MODEL OF DYNAMICS OF A RIGID DISK ON A FLEXIBLE SHAFT

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The self-oscillatory solutions of the mathematical model of transverse vibrations of a rotating horizontal flexible shaft with a solid disk proposed in [1] are investigated. It is assumed that the shaft and disc are homogeneous and perfectly balanced, the axes of the shaft and disc coincide, the ends of the shaft rest on bearings, the rotation speed is constant. The disc is mounted on a shaft at some distance from one of the ends. The shaft material is assumed to be inherently elastic and subordinate to the nonlinear rheological model of Yu.N.Robotnov [2]. The mathematical model is an initial boundary value problem for a system of two nonlinear partial differential equations and an infinite (integral) delay of the argument. The boundary conditions contain higher time derivatives and nonlinear lagging functionals. The mathematical model describes the dynamics of the middle line of a flexible shaft. For the initial boundary value problem, the concept of a generalized solution is defined, its existence, uniqueness and continuous dependence on initial conditions and parameters are proved. The stability of the zero solution (stable rotation) of the initial boundary value problem is investigated. In the plane of the main parameters (rotational velocity and linear coefficient of external friction), using the D-partition method, the stability (instability) regions of the zero solution of the initial boundary value problem are constructed. The possibility of loss of stability of the zero solution is shown, due to the passage through the imaginary axis of the complex plane of one, two and three pairs of complex conjugate points of the spectrum of the characteristic beam of operators. The bifurcations of self-oscillating solutions in these cases of loss of stability of the zero solution are investigated. The theory of central manifolds of distributed dynamical systems is used as a research method, which makes it possible to reduce the study of bifurcations of self-oscillatory solutions of a distributed system to solving a similar problem for systems of ordinary differential equations describing the behavior of trajectories on critical central manifolds. The possibility of bifurcation of periodic solutions (direct circular precession), invariant tori (beating modes) and more complex self-oscillatory solutions is shown. Asymptotic formulas are constructed for self-oscillating solutions.

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EXISTENCE, STABILITY AND NUMBER OF INVARIANT MANIFOLDS OF A PERIODIC BOUNDARY VALUE PROBLEM FOR NONLINEAR FUNCTIONAL PARTIAL DIFFERENTIAL EQUATION

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We consider a periodic boundary value problem (BVP)

$$w_t = aw_{xx} - by_x - (y^2)_x, \tag{1}$$

$$w(t, x + 2\pi) = w(t, x) \tag{2}$$

where $y = y(t, x) = w(t, x - h), a, b, h$ are some positive constants. Nonlinear functional differential equation (1), (2) is one of the variants of the equation, which in a number of sections physics of boundary phenomena is called the "nonlocal erosion equation" [1-3].

A characteristic feature of BVP (1),(2) is the following:

1) it can be included in the class of abstract parabolic equations;

2) has a one-parameter family of spatially homogeneous equilibrium states $u(t,x) = \alpha$;

3) $M_0(w) = \frac{1}{2\pi}\int\limits_0^{2\pi} w(t,x)dx = \alpha$, i.e. spatial the average over the variable x does not depend on t (can be interpreted as the first BVP integral).

For BVP (1), (2) it is possible to show the existence of a positive constant ε_0, such that for all $\varepsilon \in (0,\varepsilon_0)$ and the corresponding choice of h the following statements are true:

1) there is a set $\{\alpha_j\}, j = 1,\ldots,m(\varepsilon_0)$, $\lim\limits_{\varepsilon_0\to\infty} m(\varepsilon_0) = \infty$ such that in the neighborhood of each equilibrium state $u(t,x) = \alpha_j$ BVP (1),(2) has a two-dimensional invariant manifold V_j;

2) each V_j is formed by t periodic solutions of the BVP, for which asymptotic formulas can be found;

3) let $k(\varepsilon_0)$ be the number of saddle invariant manifolds V_j. Then $\lim\limits_{\varepsilon_0\to 0} k(\varepsilon_0) = \infty$.

From the "formal" point of view, BVP (1),(2) has a countable number of saddle two-dimensional invariant manifolds filled with saddle periodic decisions.

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PERIODIC SOLUTIONS OF THE TODA RELAY CHAIN

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Let's consider a system of nonlinear equations describing the dynamics of intercoupled nonlinear oscillators. In general, the system has the form [1]

$$M\ddot{r}_j = f(r_{j+1}) + f(r_{j-1}) - 2f(r_j). \tag{1}$$

Here $r_j(t)$ denotes the position of the j-th spring relative to its equilibrium length, and the function f characterises the tension force of the spring.

This type of system was first studied by Morikazu Toda in the context of investigating the dynamic behaviour of elements in a crystal lattice. In his works [2,3], he explored solutions to the equation for $f(r) = -\alpha(1 - e^{-\beta r})$. In this paper, we consider $f(r)$ as a piecewise constant function, defined by the following formula

$$f(r) = \begin{cases} \alpha, & r > 0, \\ 0, & r = 0, \\ -\beta, & r < 0, \end{cases} \tag{2}$$

where α and β are positive parameters.

For a system (1) of $m+1$ identical oscillators arranged in a ring, a smooth unstable periodic solution is constructed in the form of a discrete travelling wave.
This means that all components are represented by the same periodic function $r(t)$ with a shift multiple of some parameter Δ

$$r_j = r(t + j\Delta), \; j = 0, 1, \ldots, m. \tag{3}$$

The technique for constructing such solutions is considered in [4,5]. The geometrical description of the phase trajectories is also described.

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USING PARABOLIC EQUATIONS WITH MODULAR TYPE NONLINEARITY IN APPLICATIONS

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Solutions of the front type of parabolic equations are often used to model various processes in problems of biophysics, chemical kinetics, sociology and economics. Similar models containing cubic nonlinearity are well known, such as the laminar flame model proposed by Zeldovich and Frank-Kamenetsky, the FitzHugh-Nagumo model of excitation propagation in the myocardium. However, solutions of the front type can also have parabolic equations with the so-called modular nonlinearity, in particular, equations of the form

$$\frac{\partial u}{\partial t} - D\Delta u = \begin{cases} f^{-}(u, \mathbf{x}), & u \le 0, \\ f^{+}(u, \mathbf{x}), & u > 0. \end{cases} \quad (\mathbf{x}, t) \in \mathcal{D}.$$

The existence of smooth solutions of the front type for initial-boundary value problems for similar equations was investigated in [1,2]. Also in these works, expressions for the front propagation velocity were obtained. In [3], conditions for the existence of stationary solutions for systems of equations containing modular nonlinearity were formulated.

In the case the front propagates in a medium with continuous sources described by cubic nonlinearity, the occurrence of stationary distributions is associated with inhomogeneities of the medium. However, there are processes in which the unperturbed medium is homogeneous, and the stationary distribution occurs at the boundary of the unperturbed medium and the one perturbed by the front passage through it. Such a situation can arise, for example, in models of combustion or tumor growth. For such problems, the use of models with modular nonlinearities turns out to be very successful. This study is devoted to the development of such models.

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ON THE INFLUENCE OF BOUNDARY CONDITION COEFFICIENTS ON THE DYNAMIC PROPERTIES OF THE LOGISTIC EQUATION WITH DELAY AND DIFFUSION

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We consider the boundary value problem

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} - r(1-u(x,t))u(x,t-1), \quad 0 \le x \le 1, \quad r>0, \quad d>0, \tag{1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \kappa u|_{x=0}\,, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=1} = \gamma u|_{x=1}\,. \tag{2}$$

Note that $u(x, t+s) \in W^2_{2\,[0,1]} \times C_{[-1,0]}$, where $s \in [-1, 0]$. The boundary value problem (1), (2) has a clear biological meaning. It describes, for example, the change in the population size in the case when through boundaries, migration is possible proportional to the population density at the corresponding boundary of the habitat. This migration is determined by the coefficients κ and γ.

It is shown that negative values of the parameter γ and positive values of κ expand the range of variation of the values of the parameter r, at which the zero equilibrium state in (1), (2) is stable, and positive γ and negative κ – narrow.

The limiting values of the parameter r are obtained, at which the zero equilibrium state is stable.

In cases close to critical and the limiting case, at $\gamma \to -\infty$ and $\kappa \to \infty$, in the problem of stability of the zero solution, an analysis of the local dynamics of the boundary value problem (1), (2) is given.

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NONLOCALITY, INTEGRABILITY, AND SOLITONS

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We will explore integrable models that include involution points. By transforming classical Lax pairs, we create nonlocal integrable models, which possess infinitely many conservation laws and symmetries. Their solitons can be obtained using Darboux transformations or by solving reflectionless Riemann-Hilbert problems in a nonlocal context.

LOCAL DYNAMICS OF A SYSTEM OF THREE AUTOGENERATORS WITH AN ASYMMETRIC CONNECTION

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Consider a system of three ring-coupled generators with asymmetric nonlinearity:

$$\ddot{u}_j + \frac{d}{dt}\,(\varepsilon u_j + \alpha u_j^2 + u_j^3) + u_j + \mu\, g(u_{j-1}) = 0\,, \quad j = 1, 2, 3\,, \qquad (1)$$

where $u_0 = u_3$, $\mu > 0$, and the sign of the parameters ε and α is arbitrary. The coupling function $g(u)$ is given by the equality

$$g(u) = u \exp\left(-\frac{u^n}{nb^n}\right), \qquad (2)$$

where $b > 0$ is fixed.

For sufficiently small values of ε and $\mu = \nu\varepsilon$ the local theory applies to the system (1). For $\mu = \varepsilon = 0$ we have the critical case of three pairs of purely imaginary roots. To find the normal form of the system (1) standard replacement

$$u_j = \sqrt{\mu}(z_j(\tau)\exp(it) + \bar{z}_j(\tau)\exp(-it)) + \mu u_{j2}(t,\tau) + \mu^{3/2} u_{j3}(t,\tau) + \ldots,$$

$j = 1, 2, 3$, was used. Here $z_j(\tau)$ — complex-valued of the slow-time $\tau = \mu t$ functions. From the conditions of solvability of problems for $u_{j3}(t,\tau)$ in the class 2π–periodic by t functions at the third step of the algorithm, the following normal form is obtained:

$$z_j' = -\nu\frac{z_j}{2} + \frac{i}{2} z_{j-1} + d z_j |z_j|^2, \quad j = 1, 2, 3, \qquad (3)$$

where $d = -\frac{3}{2} + \frac{2\alpha^2}{3} i$.

A complete bifurcation analysis is performed for the constructed normal form. An important role of the nonlinear coupling between partial systems is shown. The specified result is obtained for the case $\alpha \neq 0$. At sufficiently small values of the parameters ε and

μ it is shown that the system (1) with a coupling function of the form (2) has an orbitally asymptotically stable cycle branching from the equilibrium state. The main difference between the cases of an asymmetric characteristic of a nonlinear element and the case of $\alpha = 0$ is that there are no symmetric or self-symmetric attractors. The dependence of the system's dynamics on the value of α is analyzed.

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INVESTIGATION OF LOCAL DYNAMICS IN THE VICINITY OF THE EQUILIBRIUM STATE OF A LOGISTIC EQUATION WITH NON-CLASSICAL BOUNDARY CONDITIONS

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Logistic equation with delay and diffusion:

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + r[1 - u(t - T, x)]u, \quad 0 \le x \le 1 \tag{1}$$

and with classical boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \ \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0 \tag{2}$$

arises in problems of mathematical ecology. In (1) $u(t, x)$ — normalized population size (density), $d > 0$ — diffusion coefficient (or mobility of the species), $r > 0$ — Malthusian coefficient, $T > 0$ — time delays that are associated with the age of puberty.

One of the most important questions regarding the model is (1), (2) is a question about the dynamics of solutions in the neighborhood of a single equilibrium state. In this regard , another form of writing this boundary value problem is usually considered in the form

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} - ru(t-T,x)[1+u], \tag{3}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \ \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0, \tag{4}$$

In this paper, the equations (3) is investigated under non-classical boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \ \left.\frac{\partial u}{\partial x}\right|_{x=1} = \alpha u(t,x_0), 0 \le x_0 < 1. \tag{5}$$

These boundary conditions also have a biological meaning: migration from the area [0, 1] and into it through the right boundary $x = 1$ depends on the population density at some intermediate point $x_0 \in [0,1)$. The problem of allocation is posed for fixed d, T and r in the plane of parameters (x_0, α) sets Ω, for elements (x_0, d) from which the zero equilibrium state of the boundary value problem (3), (5) stable.

The research is based on the use of standard numerical methods.

For this reason, we consider boundary conditions of another type, "similar" to (5), but differing from (5) in that one boundary condition involves delay:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \ \left.\frac{\partial u}{\partial x}\right|_{x=1} = \alpha u(t-h,x_0), 0 \le x_0 < 1, 0 < h < T. \tag{6}$$

For the boundary value problem (3), (6), we can use general results on the existence and uniqueness of solutions. The stability of both boundary value problems, (3), (5) and (3), (6), was analyzed by numerical methods.

Here are the main results. Figures 2 show two curves $\alpha_{\pm}(x_0)$ for $d = 0.1$, $T = 1$ and for two different values of the r parameters, which define the boundaries of the Ω domain. Null solution (3), (5) for parameters satisfying the conditions

$$\alpha_-(x_0) < \alpha < \alpha_+(x_0) \tag{7}$$

is stable, and when

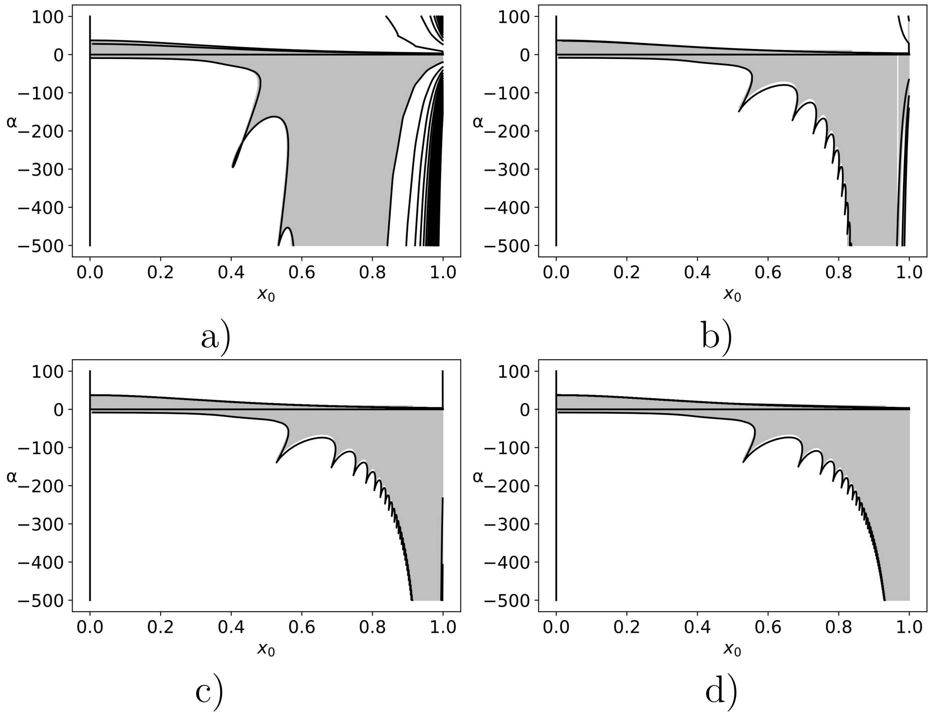

Fig. 1: The stability area of the zero solution is highlighted in gray (3), (6). Parameter values: $T = 1$, $r = 1$, $d = 10^{-1}$ and a) $h = 10^{-1}$, b) $h = 10^{-2}$, c) $h = 10^{-3}$, d) $h = 0$

$$\alpha < \alpha_{-}(x_0) \text{ or } \alpha > \alpha_{+}(x_0) \tag{8}$$

unstable.

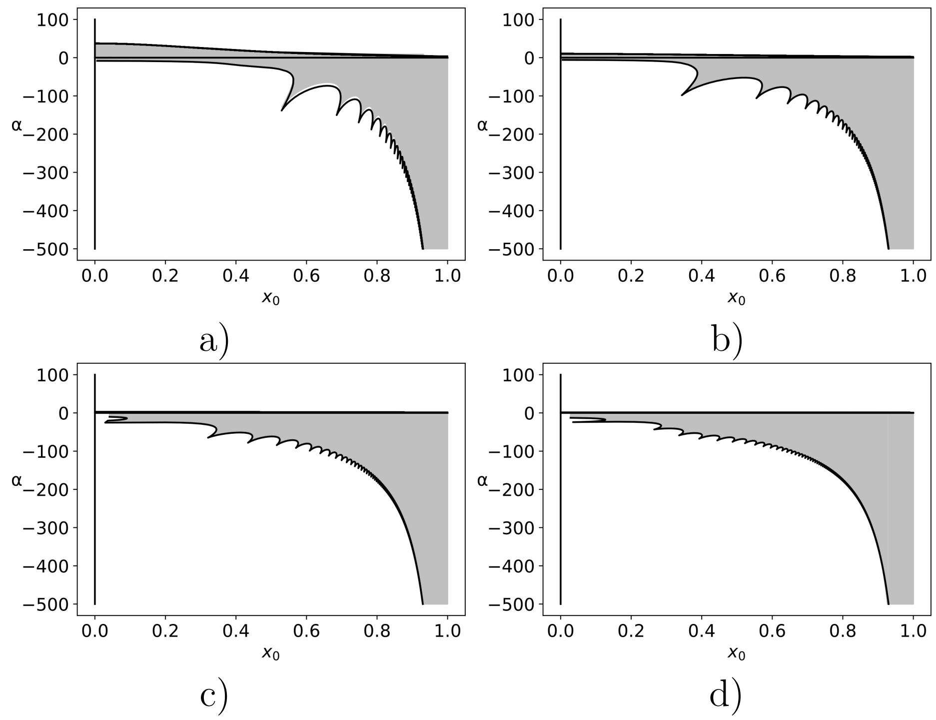

Fig. 2: Domain Ω for parameter values $T = 1$, $r = 1$ and a) $d = 0.1$ b) $d = 0.2$ c) $d = 0.5$ d) $d = 1$

THE BEHAVIOR OF SOLUTIONS TO A SECOND-ORDER DIFFERENTIAL EQUATION WITH A DELAYED IMPULSE FEEDBACK

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Consider a second-order nonlinear differential equation with a delay

$$\ddot{x} + \sigma\dot{x} + x = f(x(t-h)),$$

where $\sigma > 0$ and $h > 0$. With respect to the nonlinear function $f(x)$, we assume that it has an impulse type: $f(x) = 0$, for $x \neq 0$ and $\int\limits_{-a}^{b} f(x)dx = f_0$ for any $a, b > 0$. Let study the behavior of solutions for fixed values of the parameters σ, f_0, h.

Define the class of initial conditions S_A, depending on the parameter A and consisting of piecewise continuously differentiable functions $\varphi(t)$ defined on the segment $[-h, 0]$ for which $\varphi(t) \neq 0$ for

$t < 0$, $\varphi(0) = 0$ and $\dot{\varphi}(0) = A$. For each initial function from $S(A)$, using the step-by-step integration method, we construct the solution $x_A(t)$ and find its first positive root $t_* = t_*(A) : x_A(t_*(A)) = 0$. If the condition $t_*(A) > h$ is fulfilled, then $x_A(t_*(A) + t) \in S(\overline{A})$, where $\overline{A} = p(A) = \dot{x}_A(t_*(A))$. Thus, the mapping $A_{n+1} = p(A_n)$ is determined, the dynamics of which describes the behavior of solutions to the initial differential equation with a delay. The specific type of display and its properties depend on the parameters σ, f_0, h.

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SUPERSYMMETRIC SOLUTIONS OF THE ASSOCIATIVE YANG-BAXTER EQUATION

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We propose a trigonometric solution of the associative Yang-Baxter equation related to the queer Lie superalgebra which in its turn satisfies the quantum Yang-Baxter equation.

ON INFINITE TOWERS OF SYMPLECTIC AND CONTACT NILMANIFOLDS

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We consider infinite towers of symplectic and contact nilmanifolds corresponding to narrow infinite-dimensional graded Lie algebras. One of the most important examples is the tower of nilmanifolds corresponding to the positive part W^+ of the Witt algebra W considered in [1]. Another important example of an infinite chain of nilmanifolds can be obtained from the infinite-dimensional Vergne algebra $\mathfrak{m}_0$ [2]. Such towers are constructed using the inductive procedure of successive one-dimensional central extensions of the corresponding finite-dimensional (nilpotent) Lie algebras. We discuss examples of nilmanifolds that cannot be included in infinite towers of this type [3,4].

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QUANTIFICATION AND COMPARISON OF MAGNETIC AND KINETIC CHAOS IN TOROIDAL PLASMAS

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The presence of non-axisymmetric perturbations of axisymmetric toroidal magnetic field results in the chaoticity of the magnetic field lines and strongly affects the charged particle motion and, therefore, the particle energy and momentum transport in toroidal plasma [1, 2]. Particle chaoticity is determined by resonance conditions relating the unperturbed orbital frequencies of the particles with the toroidal and poloidal numbers of the perturbative modes [3]. The Guiding Center (GC) motion [4] of low-energy particles approximately follows the magnetic field lines so that magnetic and kinetic chaos have similar characteristics. However, higher-energy particles may undergo large drifts across the magnetic field lines, and the chaotic characteristics of their GC motion can be quite different from those of the underlying magnetic field. Here we present the outcomes of a systematic comparison of magnetic and kinetic chaos [4] based on the utilization of the Smaller Alignment Index (SALI) [5, 6], which is an efficient chaos detection technique. The efficient quantification of chaos by the SALI method enables the assignment of a value characterizing the chaoticity of each orbit in the space of the three constants of the motion, namely, energy, magnetic moment, and toroidal momentum. In this way, we construct detailed colour plots, which provide a unique overview of the different effects of a specific set of perturbations on the entire range of trapped and passing particles, as well as the radial location of the chaotic regions. Our approach constitutes a valuable method for the study of the chaotic behaviour of toroidal fusion devices.

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HYPERBOLIC SYSTEMS: SPECIAL TRANSFORMATIONS AND EXACT SOLUTIONS

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In this paper we propose a method for constructing exact solutions of nonlinear hyperbolic systems of two first-order partial differential equations with two independent variables.

Such systems are represented by a pair of differential 2-forms on the four-dimensional space $\mathbb{R}^4$ with coordinates x_1, x_2, u_1, u_2. Here x_1, x_2 are independent variables and u_1, u_2 are unknown functions.

This system defines an almost product structure on the space $\mathbb{R}^4$ (see [1]). Let A be a linear operator corresponding to this structure. If the Nijenhuis bracket of this operator is zero, then the equation is reduced to the wave system by changing variables [2]. Therefore, it can be solved exactly.

However, the condition of equality to zero of the Nijenhuis bracket is often not satisfied and this method does not work.

In the report, we consider exactly this case. The main idea of our method is as follows.

If the system has a conservation law for which the non-degeneracy condition is satisfied, then such a system can be written as a single second-order equation $\mathcal{E}$ [1]. According to the results of V.V. Lychagin [3], this equation can be considered as a differential 2-form on the five-dimensional space of 1-jets $J^1(\mathbb{R})$.

A.G. Kushner showed (see [4]) that two differential 2-forms λ_+, λ_- on this space can be invariantly associated with this equation (he called them the Laplace forms). These forms are tensor analogues of the Laplace semi-invariants for linear hyperbolic equations. If these forms are identically equal to zero, then the equation $\mathcal{E}$ is contact equivalent to the wave equation [4]. The general solution of the wave equation is known. Applying the inverse transformation to this solution, we obtain the solution of equation $\mathcal{E}$. Note that this solution may be multi-valued. Knowing it, we can construct an exact (multi-valued) solution of the original system.

So, the essence of the method consists in replacing the transformations of the four-dimensional space with special transformations of a special five-dimensional space.

We illustrate our method for the Barenblatt system which describes the process of two-phase filtration (for example, oil and water) in the presence of surface-active reagents [5]:

$$\begin{cases} s_t + H_x = 0, \\ (cs + \phi(1-s) + a)_t + (cH + (1-H)\phi)_x = 0. \end{cases}$$

where $s(t,x)$ is a water saturation, $c(t,x)$ is a relative concentration of active reagent in water, $H(s,c)$ is the Buckley–Leverett function, $\phi(c)$ is a concentration of active reagent in oil, $a(c)$ is a concentration of active reagent deposited on the pores, t is the time, x is the spatial variable (the x-axis coincides with the direction of movement of the fluid). Here independent variables are t, x, unknown functions are s, c. The functions H, ϕ, a depend on physical properties of the pore space and chemical properties of active reagent.

The initial system in the four-dimensional space $\mathbb{R}^4(t,x,s,c)$ can be represented by a pair of differential 2-forms (see [6])

$$\begin{cases} \omega_1 & = -(H_s(\phi'(s-1) - s - a') + (\phi'-1)H - \phi')H_c\chi^{-1}\, dt\wedge dc \\ & +2H_c(\phi'(s-1) - a' - s)\chi^{-1} dx\wedge dc + dx\wedge ds - H_s\, dt\wedge ds, \\ \omega_2 & = H_s dt\wedge ds + H_c dt\wedge dc - dx\wedge ds. \end{cases} \tag{1}$$

Here $\chi = H_s(\phi'(s-1) - s - a') + H(1-\phi') + \phi'$.

Applying the transformation

$$\Phi_1 \colon \{q_1 = H_s t - x, \quad q_2 = H_c t, \quad p_1 = s, \quad p_2 = c\}$$

to forms (1), we get the equation $\mathcal{E}$:

$$H_{p_2}(\mu H_{p_1} - \eta)v_{q_1q_2} + \mu H_{p_2}v_{q_2q_2} + (\mu H_{p_1p_1}H_{p_2} + H_{p_1p_2}(\mu H_{p_2} - \eta))q_2(v_{q_1q_2}^2 - v_{q_1q_1}v_{q_2q_2}) = 0,$$

where $\mu = (p_1 - 1)\phi' - p_1 - a'$ and $\eta = (\phi' - 1)H - \phi'$.

If both Laplace 2-forms are zero, then the corresponding nonlinear partial differential equation can be reduced to a wave equation using nonlinear contact transformations. If one Laplace invariant 2-form is zero and the other form is not, then the equation of the second order is reduced to a linear one by contact transformations.

Conditions are found under which both Laplace forms vanish. For such cases, we construct exact solutions of the Barenblatt equation.

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TIKHONOV-TYPE SYSTEMS IN THE CASE OF EXCHANGE OF STABILITIES

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We consider the problem

$$\varepsilon^2 \Delta u - \frac{\partial u}{\partial t} - g(u, v, x, \varepsilon) = 0,$$
$$\Delta v - \frac{\partial v}{\partial t} - f(u, v, x, \varepsilon) = 0, \ x \in D \subset R^n, \ t \in R^+,$$

with corresponding boundary and initial conditions in the case when the isolation of the roots $\varphi_1(v, x)$ and $\varphi_2(k(x), x)$ of the degenerate equation is violated on the surface $v = k(x)$:

There is a continuous function $k(x)$: $D \to R$ such that $\varphi_1(k(x), x) = \varphi_2(k(x), x)$ for $x \in D$, and the exchange in stability occurs on the closed (n-1)-dimensional hypersurface $\Gamma(x) \in D$, dividing D into outer D^- and inner D^+ subdomains with respect to $\Gamma(x)$.

The composite stable solution is determined. The proof of the theorem of the existence of the stable stationary solution is proved, its asymptotic approximation is given.

The results are extended to analogous periodic parabolic boundary problem in the case of the exchange in stability.

The results presents a further development of the results presented in the review [1].

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THE ELLIPTIC LATTICE KDV SYSTEM REVISITED, AND ASSOCIATED ELLIPTIC DISCRETE INTEGRABLE SYSTEMS

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The elliptic lattice KdV system was introduced in [1] as a natural multi-component generalisation of the lattice potential KdV equation associated with an elliptic curve. It exhibits many of the expected integrability features: a Lax representation, multi-soliton solutions and integrable reductions. However, compared to the usual lattice potential KdV equation (H1 of the ABS list) it lacks a number of important features: the Lax matrices no longer factorise, there is no Lagrangian structure known (so far) and the simplest nontrivial periodic reductions are already of higher genus (compared to g=1 for the usual case). Relation to the well-known Q4 or Adler's equation of [2], or to any other quadrilateral lattice equations, are yet to be estabished. In the present note, I will present some novel results on this elliptic lattice KdV system: a coupled set of multiquadratic equations (in the sense of [3]), and an associated elliptic Yang-Baxter map. Furthermore, there is connection with an elliptic lattice AKP system, given in [4], which is in contrast to the elliptic B type KP system, [5], which first arose as a byproduct of some lattice systems of B type by Date, Jimbo and Miwa in 1983.

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WEAK INNER LAYER IN THE REACTION-DIFFUSION-ADVECTION PROBLEM IN THE CASE OF REACTION DISCONTINUITY

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The present work is devoted to the study of a one-dimensional reaction-advection-diffusion equation with weak smooth advection and a discontinuous reaction in the spatial coordinate:

$$\begin{cases} N_\varepsilon u := \varepsilon^2 \dfrac{\partial^2 u}{\partial x^2} - \varepsilon a(u,x)\dfrac{\partial u}{\partial x} - f(u,x) - \varepsilon f_1(u,x) = 0, \quad -1 < x < 1, \\ f_1(u,x) := \begin{cases} f_1^{(+)}(u,x), & u \in I_u, \quad x_p < x \le 1, \\ f_1^{(-)}(u,x), & u \in I_u, \quad -1 \le x < x_p, \end{cases} \\ f(u,x) := \begin{cases} f^{(+)}(u,x), & u \in I_u, \quad x_p < x \le 1, \\ f^{(-)}(u,x), & u \in I_u, \quad -1 \le x < x_p, \end{cases} \\ \dfrac{\partial u}{\partial x}(-1,\varepsilon) = 0, \quad \dfrac{\partial u}{\partial x}(1,\varepsilon) = 0. \end{cases} \tag{1}$$

Here ε – a small parameter, $0 < \varepsilon < \varepsilon_0 \ll 1$, $x_p \in (-1;1)$, I_u - a segment of varying of function $u(x,\varepsilon)$.

The construction of asymptotics, proof of existence and study of stability of stationary solutions with the constructed asymptotics, possessing a weak internal layer, which is formed near the discontinuity point, are carried out for problem (1). To construct the asymptotics, the method of Vasilyeva A.B. was used, to justify the existence of the solution – the asymptotic method of differential inequalities, to study the stability – the method of contracting barriers. It is shown that such a solution as a solution of the corresponding initial-boundary value problem is asymptotically stable in the sense of Lyapunov. A stability region of finite (not asymptotically small) width for such a solution is indicated and it is established that the solution to the stationary problem is unique in this region.

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CORANK-1 SINGULARITIES OF TYPICAL INTEGRABLE SYSTEMS WITH 3 DEGREES OF FREEDOM

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We obtain a classification (up to fiberwise diffeomorphism) of corank-1 singularities of typical real-analytic integrable systems on 4- and 6-dimensional symplectic manifolds. We prove that, near a corank-1 orbit, the Liouville foliation of such a system is diffeomorphic to the standard model. We also depict phase portraits and their bifurcations near such singularities, and prove structural stability of these singularities. Our singularities in the 4D case are parabolic orbits with resonances [2], and bifurcations of such orbits in the 6D case. Let us proceed with precise statements.

An integrable Hamiltonian system on a $2n$-dimensional symplectic manifold (M, ω) is given by a smooth mapping

$$\mathcal{F} = (f_1, \ldots, f_n) : \ M \longrightarrow \mathbb{R}^n,$$

where $\{f_i, f_j\} = 0$. A *singular Lagrangian foliation* on (M, ω) arises whose fibers are connected components of the sets $\mathcal{F}^{-1}(c)$, $c \in \mathbb{R}$. Regular compact fibers are Liouville tori. Suppose that M is compact. Then the mapping $\mathcal{F}$ generates a Hamiltonian $\mathbb{R}^n$-action on M.

Definition. A semilocal singularity (i.e., a compact $\mathbb{R}^n$–orbit) of a real-analytic integrable system $(M, \omega, \mathcal{F})$ is called *structurally stable* if it has a neighborhood $U^{\mathbb{C}}$ in $M^{\mathbb{C}}$ such that any real-analytic integrable system $(\overline{U^{\mathbb{C}}}, \widetilde{\omega}^{\mathbb{C}}, \widetilde{\mathcal{F}}^{\mathbb{C}})$ sufficiently close to the given system in $U^{\mathbb{C}}$ w.r.t. C^∞-topology can be represented as follows: $\widetilde{\mathcal{F}}^{\mathbb{C}} = \phi^{\mathbb{C}} \circ \mathcal{F}^{\mathbb{C}} \circ \Phi^{\mathbb{C}}$, where $\Phi^{\mathbb{C}} : \overline{U^{\mathbb{C}}} \to M^{\mathbb{C}}$ and $\phi : \mathbb{C}^n \to \mathbb{C}^n$ are an embedding and a homeomorphism, resp., close to the identities. If Φ is a diffeomorphism onto its image, the singularity is called *smoothly structurally stable.*

Consider the class $\mathcal{S} = \mathcal{S}(M)$ of integrable systems on M (called semitoric) for which the functions $f_2, \ldots, f_n$ generate a locally-free Hamiltonian action of the $(n-1)$-torus on M [4]. Let $\mathcal{O}$ be a corank-1 semilocal singularity, i.e., a compact $\mathbb{R}^n$-orbit such that $\operatorname{rk} d\mathcal{F}(\mathcal{O}) = n-1$.

Definition. We define the *standard model* $(M_{st}, \omega_{st}, \mathcal{F}_{st})$ by setting

$$M_{st} = (D^2 \times D^{n-1} \times T^{n-1})/G, \quad \omega_{st} = dx \wedge dy + \sum_{i=1}^{n-1} dI_i \wedge d\varphi_i,$$

$$\mathcal{F}_{st}(\mathbf{z}, I, \varphi) = (H_{s,k,\alpha(I)}(\mathbf{z}, I'), I),$$

where G is a finite cyclic subgroup of $SO(2)$, $H_{s,k,\alpha(I)}(x, y, I') = H(x, y, I_1, \ldots, I_k)$ is a smooth G-invariant function of variables $\mathbf{z} = (x, y) \in D^2$ and parameters $I = (I_1, \ldots, I_{n-1}) \in D^{n-1}$, $\varphi = (\varphi_1, \ldots, \varphi_{n-1}) \in T^{n-1}$, $s = |G| \in \mathbb{N}$, $1 \le k < n \le 3$, and the action of the group G on the direct product has the form $(x, y, I, \varphi) \mapsto (x \cos(2\pi\ell/s) - y \sin(2\pi\ell/s), x \sin(2\pi\ell/s) + y \cos(2\pi\ell/s), I, \varphi_1 + 2\pi/s, \varphi_2, \ldots, \varphi_{n-1})$, $0 \le \ell < s$, $(\ell, s) = 1$. Here $\alpha(I)$ is a smooth function.

Thus, the standard model is the semitoric integrable system $(M_{st}, \omega_{st}, \mathcal{F}_{st})$ with a Hamiltonian $(\mathbb{R} \times T^{n-1})$-action and a compact $(n-1)$-dimensional orbit $\mathcal{O}$ given by $x = y = I_1 = \cdots = I_{n-1} = 0$ having multipliers $e^{\pm \frac{2\pi \ell i}{s}} = \xi^{\pm \ell}$ (with $\xi = e^{2\pi i/s}$), twisting resonance $\ell/s \mod 1$ (see [3]) and resonance order $s = |G|$.

For any $s \in \mathbb{N}$, consider the cyclic group $G \subset SO(2)$ of order s, see above, the G-invariant Morse functions $H_{s,0} = H_{s,0}^{\pm,\pm}(x, y) = \pm |z|^2 = \pm(x^2 + y^2)$ and $H_{s,0} = H_{s,0}^{+,-}(x, y) = x^2 - y^2$ (for $s = 1, 2$ only), and two families of $\mathbb{Z}_s$-invariant functions $H_{s,k,\alpha}(\mathbf{z},\varepsilon)$, $k=1,2$:

$$H_{s,1,\alpha}(\mathbf{z},\varepsilon)=\begin{cases} \pm x^2+y^{s+2} & +\varepsilon y^s, \quad s=1,2,\\ \mathrm{Re}(z^3) & +\varepsilon|z|^2, \quad s=3,\\ \mathrm{Re}(z^4)+\alpha|z|^4 & +\varepsilon|z|^2, \quad s=4,\ \alpha^2\neq 1,\\ \mathrm{Re}(z^s)+|z|^4 & +\varepsilon|z|^2, \quad s\geq 5,\end{cases}$$

$$H_{s,2,\alpha}(\mathbf{z},\varepsilon)=\begin{cases} \pm x^2+y^{2s+2} & +\varepsilon_2 y^{2s}+\varepsilon_1 y^s, \quad s=1,2,\\ \mathrm{Re}(z^4)\pm|z|^4+|z|^6 & +\varepsilon_2|z|^4+\varepsilon_1|z|^2, \quad s=4,\\ \mathrm{Re}(z^5)+|z|^6 & +\varepsilon_2|z|^4+\varepsilon_1|z|^2, \quad s=5,\\ \mathrm{Re}(z^6)+\alpha|z|^6+|z|^8 & +\varepsilon_2|z|^4+\varepsilon_1|z|^2, \quad s=6,\ \alpha^2\neq 1,\\ \mathrm{Re}(z^s)+|z|^6+\alpha|z|^8 & +\varepsilon_2|z|^4+\varepsilon_1|z|^2, \quad s\geq 7.\end{cases}$$

Here $z=x+iy$, $\varepsilon=(\varepsilon_i)\in\mathbb{R}^k$ is a small parameter, $\alpha\in\mathbb{R}$ is a parameter called *modulus*.

Theorem 1 ([5]). *Let $n=\frac{1}{2}\dim M\in\{2,3\}$ and $\mathcal{S}_{st}\subseteq\mathcal{S}$ denote the class of systems all of whose local singularities are fiberwise diffeomorphic to standard ones. Then $\mathcal{S}_{st}$ is open and dense in $\mathcal{S}$ with respect to the C^∞-topology. Furthermore, any singularity that is fiberwise diffeomorphic to the standard model, in which the modulus is present (respectively, absent), is structurally stable (respectively, smoothly structurally stable) with respect to small real-analytic integrable perturbations.*

Remark. Theorem 1 describes parabolic trajectories with resonances [2] for $n=2$, and their generic bifurcations for $n=3$.

Theorem 1 does not follow from the classification in [1].

Remark. The classification of corank-2 singularities of typical integrable Hamiltonian systems with three degrees of freedom was recently obtained by E.A. Kudryavtseva and L.M. Lerman in [6], and we aim to prove structural stability of these singularities (and thereby extend Theorem 1 to such singularities).

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EXISTENCE AND STABILITY OF STATIONARY SOLUTIONS WITH BOUNDARY LAYER IN A TWO-DIMENSIONAL SYSTEM OF FAST AND SLOW REACTION-DIFFUSION-ADVECTION EQUATIONS WITH KPZ NONLINEARITIES

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We consider a system of fast and slow reaction-diffusion-advection equations with KPZ nonlinearities:

$$\begin{aligned} &\mathcal{N}_u(u,v) := \varepsilon^2\Delta u - \varepsilon^2 A(u,x)\,(\nabla u)^2 - g(u,v,x,\varepsilon) = 0, \quad x = (x_1,x_2)\in D,\\ &\mathcal{N}_v(u,v) := \Delta v - B(v,x)\,(\nabla v)^2 - f(u,v,x,\varepsilon) = 0, \end{aligned} \qquad (1)$$

Here D is a simply connected area on the plane (x_1, x_2) with a smooth simple boundary Γ, ε is a small parameter lying in the range $(0;\varepsilon_0]$, $\varepsilon_0 > 0$.

PDE equations with nonlinearities involving the scalar square of the unknown function gradient (known as Kardar–Parisi–Zhang (KPZ) nonlinearities) arise in various applications: population dynamics, free surface growth in polymer theory, nonlinear theory of thermal conductivity.

For this problem, conditions are obtained under which solutions with a boundary layer are Lyapunov stable. The asymptotic method of differential inequalities is used to prove the existence and stability theorems. The boundary layer asymptotics of solutions are constructed in the case of Neumann and Dirichlet boundary conditions. The case of quasimonotone sources and systems without the quasimonotonicity requirement is also considered.

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BI-HAMILTONIAN STRUCTURES OF DUBROVIN–NOVIKOV TYPE

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In this talk we consider a classification of bi-Hamiltonian structures of Dubrovin–Novikov type based on a number of common flat coordinates. Also we present a list of most remarkable integrable hydrodynamic type systems, equipped by such pairs of local Hamiltonian structures.

IS CLASSICAL INTEGRABLE SYSTEM A BIT QUANTUM?

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Existence of such quantum effects as creation/annihilation of particles and dark energy is proved to arise in the study of known models of classical mechanics, such as the Calogero–Moser and Ruijsenaars–Schneider systems.

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PARAMETERS ESTIMATION IN THE TRAFFIC FLOW MODEL

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This study develops an extension of the leader-follower mathematical model, which describes the movement of $N \in \mathbb{N}$ vehicles, offering a modified and improved version of the model proposed in [1,2]. In the original model, each vehicle n follows the movement of the vehicle immediately ahead, $n-1$. However, in this paper, the n-th vehicle not only considers the movement of the $n-1$ vehicle but also forecasts its behavior by taking into account the movement of the vehicle two positions ahead, $n-2$. The n-th vehicle then adapts its own movement based on this forecast, meaning that the driver's behavior is influenced by both their own actions and the actions of the vehicle that is two vehicles ahead.

Vehicles with numbers $n = 1$ and $n = 2$ cannot forecast the movement of other vehicles as there are no vehicles ahead of them. Consequently, they move according to the model proposed in [1,2].

Let $y_k(t)$, where $k = n-1, n > 2$, denote the position of the bumper of the predicted vehicle at time t. Then, $\dot{y}_k(t)$ and $\ddot{y}_k(t)$ represent its speed and acceleration, respectively.

Since the movement of the $n-1$ vehicle is predicted, the driver of the n-th vehicle receives information about its behavior without delay, unlike the case where the driver simply watches the vehicle ahead and receives data with a time delay τ.

Thus, the model considering the predicted dynamics of the leading vehicle takes the following form:

$$\begin{cases} \ddot{x}_1(t) = & R_1 \left[a_1 \left(v_{max,1} - \dot{x}_1(t)\right)\right] - (1 - R_1)H_1, \\ \ddot{x}_2(t) = & R_2 \left[a_2 \left(P_2 - \dot{x}_2(t)\right)\right] - (1 - R_2)H_2, \\ \ddot{y}_k(t) = & \hat{R}_k \left[a_k \left(\hat{P}_k - \dot{y}_k(t)\right)\right] - (1 - \hat{R}_k)\hat{H}_k, \\ \ddot{x}_n(t) = & \tilde{R}_n \left[a_n \left(\tilde{P}_n - \dot{x}_n(t)\right)\right] - (1 - \tilde{R}_n)\tilde{H}_n, \\ & x_n(t) = y_k(t) = \lambda_n, \quad \dot{x}_n(t) = \dot{y}_k(t) = v_n, \quad \text{for } t \in [-\tau, 0]. \end{cases}$$

The relay functions $\hat{R}_k$ and $\tilde{R}_n$ of system are as follows:

$$\hat{R}_k = \begin{cases} 1, & \text{if } x_{k-1}(t-\tau) - y_k(t) > (\tau + \tau_b)\dot{y}_k(t) + \dot{y}_k^2(t)/2\mu g + l_k, \\ 0, & \text{if } x_{k-1}(t-\tau) - y_k(t) \le (\tau + \tau_b)\dot{y}_k(t) + \dot{y}_k^2(t)/2\mu g + l_k, \end{cases}$$

and

$$\tilde{R}_n = \begin{cases} 1, & \text{if } y_{n-1}(t) - x_n(t) > (\tau + \tau_b)\dot{x}_n(t) + \dot{x}_n^2(t)/2\mu g + l_n, \\ 0, & \text{if } y_{n-1}(t) - x_n(t) \le (\tau + \tau_b)\dot{x}_n(t) + \dot{x}_n^2(t)/2\mu g + l_n, \end{cases}$$

respectively.

The logistic function $\hat{P}_k$ is given by:

$$\hat{P}_k = \frac{v_{max,k} - \hat{V}_k}{1 + \exp[k_k(-(x_{k-1}(t-\tau) - y_k(t)) + \hat{S}_k)]} + \hat{V}_k,$$

where $\hat{V}_k = \min\left(\dot{x}_{k-1}(t-\tau), v_{max,k}\right)$, and the parameter $\hat{S}_k$ of the logistic curve is:

$$\hat{S}_k = (\tau + t_b)\dot{y}_k(t) + \dot{y}_k^2(t)/2\mu g + l_k + \tau(\dot{x}_{k-1}(t-\tau) - \dot{y}_k(t)).$$

The logistic function $\tilde{P}_n$ is given by:

$$\tilde{P}_n = \frac{v_{max,n} - \tilde{V}_n}{1 + \exp[k_n(-(y_{n-1}(t) - x_n(t)) + \tilde{S}_n)]} + \tilde{V}_n,$$

where $\tilde{V}_n = \min\left(\dot{y}_{n-1}(t), v_{max,n}\right)$, and $\tilde{S}_n$ is:

$$\tilde{S}_n = (\tau + t_b)\dot{x}_n(t) + \dot{x}_n^2(t)/2\mu g + l_n + \tau(\dot{y}_{n-1}(t) - \dot{x}_n(t)).$$

The Heaviside functions $\hat{H}_k$ and $\tilde{H}_n$ of the system are:

$$\hat{H}_k = \begin{cases} q_k\left(\dot{y}_k(t)\dfrac{\hat{\Delta}\dot{y}_k}{\hat{\Delta}y_k - l_k}\right)^2, & \text{if } q_k\left(\dot{y}_k(t)\dfrac{\hat{\Delta}\dot{y}_k}{\hat{\Delta}y_k - l_k}\right)^2 \le \mu g, \\ \mu g, & \text{if } q_k\left(\dot{y}_k(t)\dfrac{\hat{\Delta}\dot{y}_k}{\hat{\Delta}y_k - l_k}\right)^2 > \mu g, \end{cases}$$

where $\hat{\Delta}\dot{y}_k = \dot{x}_{k-1}(t-\tau) - \dot{y}_k(t)$ and $\hat{\Delta}y_k = x_{k-1}(t-\tau) - y_k(t)$, and

$$\tilde{H}_n = \begin{cases} q_n\left(\dot{x}_n(t)\dfrac{\tilde{\Delta}\dot{x}_n}{\tilde{\Delta}x_n - l_n}\right)^2, & \text{if } q_n\left(\dot{x}_n(t)\dfrac{\tilde{\Delta}\dot{x}_n}{\Delta x_n - l_n}\right)^2 \le \mu g, \\ \mu g, & \text{if } q_n\left(\dot{x}_n(t)\dfrac{\tilde{\Delta}\dot{x}_n}{\tilde{\Delta}x_n - l_n}\right)^2 > \mu g, \end{cases}$$

where $\tilde{\Delta}\dot{x}_n = \dot{y}_{n-1}(t) - \dot{x}_n(t)$ and $\tilde{\Delta}x_n = y_{n-1}(t) - x_n(t)$.

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AN UNSTABLE CYCLE WITH A "SHORT" PERIOD IN ONE RELAY DIFFERENTIAL EQUATION WITH DELAY

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Considering the relay version [1] of the generalized Hutchinson equation [2]

$$\dot{u} = \lambda F(u(t-1))u. \tag{1}$$

Here, the scalar function $u(t) > 0$ represents the normalized membrane potential, $\lambda > 0$ is the speed of electrical processes in the nerve cell,

$$F(u) \stackrel{\text{def}}{=} \begin{cases} 1, & 0 < u \le 1, \\ -a, & u > 1. \end{cases}$$

After the exponential substitution $u = e^{\lambda x}$, equation (1) takes the form of a difference-differential equation

$$\dot{x} = R(x(t-1)) \tag{2}$$

with a piecewise constant right side, where

$$R(x) \stackrel{\text{def}}{=} \begin{cases} 1, & x \le 0, \\ -a, & x > 0. \end{cases}$$

In work [2], equation (2) was considered with a negative continuous initial function; it was proven an existence and orbital stability of a periodic solution

$$x_0(t) \stackrel{\text{def}}{=} \begin{cases} t, & t \in [0,1], \\ -a(t-t_0), & t \in [1, t_0+1], \\ t - T_0, & t \in [t_0+1, T_0], \end{cases} \qquad x_0(t+T_0) \equiv x_0(t), \tag{3}$$

where

$$t_0 \stackrel{\text{def}}{=} (a+1)/a, \quad T_0 \stackrel{\text{def}}{=} (a+1)^2/a,$$

The period T_0 is longer than the delay 1.

In this work, all possible solutions with continuous initial functions that contain an arbitrary number of roots over the delay length interval are constructed. It is proven that there exists a value of the roots of the initial function for which the equation has a periodic unstable solution with a period shorter than the delay. Moreover, all such solutions are homothetic to solution (3). In cases where the zeros of the initial function are chosen differently or there are fewer than two, the regime (3) is established.

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PERIODIC SOLUTIONS OF A SINGLE DIFFERENTIAL EQUATION WITH THREE DELAYS FROM NEURODYNAMICS

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The paper considers a model of a ring chain of neurons, the functioning of each of which is described by an equation with two delays. This equation was studied in articles [1,2]. The model under study is a modification of the one considered in [3], where the single-delay equation, the generalized Hutchinson equation, lies as a model of a solitary neuron [4]. As in [3], we construct discrete traveling waves. This means that we are looking for a periodic solution to the system, such that all components coincide with the same function, shifted by a multiple of a certain parameter. To find this solution, an auxiliary differential-difference equation of the Volterra type with three delays is investigated. For this equation, the existence of a stable periodic solution containing any predetermined number of bursts per period (bursting effect in the sense of definition [1]) is established.

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ON THE CARDAN MOTION IN AN EXTENDED HYPERBOLIC PLANE

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In kinematics, the *Cardan motion* is defined as the motion of a plane α with respect to a coinciding plane β such that two points A and B of α move along two orthogonal lines l, m of β (see, for instance, [1, Section 2.3] or [2]). In Euclidean geometry, an arbitrary point of the plane α traces in general an ellipse during a Cardan motion (see [1, Theorem 2.3.1]). In particular, the midpoint of the moving segment AB describes a circle. The paper [3] of O. Bottema is most likely the first step in studying of the Cardan motion in non-Euclidean geometry. In [3] it is proved that, in general, in an elliptic plane the path of an arbitrary point of the generating line AB during the Cardan motion is a quartic curve.

We study this motion in an extended hyperbolic plane H^2, the connected components of which are the Lobachevskii plane λ_2 and the hyperbolic plane $\widehat{H}$ of positive curvature adjacent to Λ_2 along the absolute oval curve γ. In the plane H^2, for a pair of two orthogonal lines, there are three possible cases. The lines l and m can belong to a hyperbolic, elliptic, or parabolic pencil of lines. In paper [4], the paths of an arbitrary point of the generating line AB during Cardan motion are investigated in H^2 under the condition that lines l and m belong to a hyperbolic pencil. In particular, the paths of the midpoint of the segment AB are studied in detail (such curves were called *Svetlana ribbons*).

In the upcoming report, we plan to present a study of the Cardan motion in H^2, provided that the lines l and m belong to an elliptic or parabolic pencil. In particular, we will establish a connection between the paths of points in the Cardan motion and the remarkable curves of the Minkowski plane.

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HOMOCLINIC ORBITS IN MATRIX NLS-TYPE SYSTEMS

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We consider a system of coupled nonlinear Schrödinger equations with even, periodic boundary conditions, which are damped and quasi-periodically forced. Under certain conditions, we establish criteria for the existence of homoclinic orbits to a spatially independent invariant torus. We compare the analysis with rigorous numerical simulation. In the second part of the talk, we describe the full-time dynamics of modulational instability in $F = 1$ spinor Bose–Einstein condensates for the case of the integrable three-component model associated with the matrix nonlinear Schrödinger equation. We obtain an exact homoclinic solution of this model by employing the dressing method which we generalize to the case of the higher-rank projectors.

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ON SINGULARITIES OF CAUSTICS IN SPACES OF DIMENSION $n \leq 5$

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Caustic is the set of critical values of a Lagrangian map. A germ of a Lagrangian map is a germ of a sweep of a gradient mapping. By Arnold's theorem on Lagrangian singularities, simple stable Lagrangian germs are defined by versal deformations of germs of smooth functions at critical points of types A, D, E. Multisingularity of a Lagrangian map at a point of the target space is the unordered set of singularities of the mapping at the preimages of this point. We will talk about the adjacencies of multisingularities of a generic Lagrangian map into a space of dimension $n \leq 5$.

GEOMETRIC ANATOMY OF THE NONWANDERING CONTINUUM POSSESSING WADA PROPERTY

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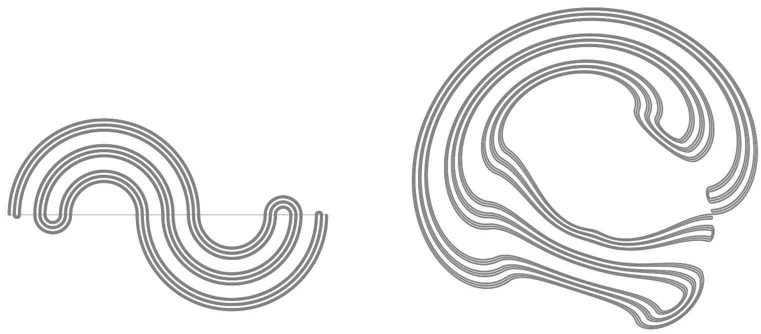
Recently, I have researched and then announced the topological classification of the Birkhoff curves and the nonwandering continua possessing Wada property. At the same time, I made a fundamental mistake by allowing the existence of more than the only fixed point belonging to the Birkhoff curve.

Theorem 1 *Birkhoff curve contains the only fixed point.*

K. Kuratowski (1928) proved that an indecomposable continuum cutting a plane into two regions turns out to be monostratic (monostratique) [1]. Therefore, the Birkhoff curve has the only fixed point with an index being equal to zero. It is simple.

So that, the Birkhoff curve is consisted to be nonwandering indecomposable continuum turning out to be two invariant regions boundary with respect to dynamic system acting on the plane. The Birkhoff curve geometric model has been constructed based on the Knaster example indecomposable continuum having two composants [2] as follows



(a) (b)

Fig. 1: (a) Knaster's continuum havitg two composants from [2]
(b) monostratic indecomposable continuum turns out to be two regions common boundary

Endpoints $(0, 0)$ and $(0, 1)$ of the Knaster's continuum are glued by the formula

$$(y - 7/20)e^{2\pi x} \mapsto x + iy.$$

Now, on the assumption of the principle of constructing the Birkhoff curve geometric model, geometric models of the nonwandering continua turning out to be three regions common boundary have been constructed as follows:

fit the first, — the indecomposable continuum having four composants has been constructed
fit the second, — now the endpoints in pairs have been glued

The continua in Fig. 3 turn out to be three regions common boundary. Moreover, these constructions turn out to be more adapted to dynamic systems (compare with the examples from [3]) than the well-known Brower and Wada examples.

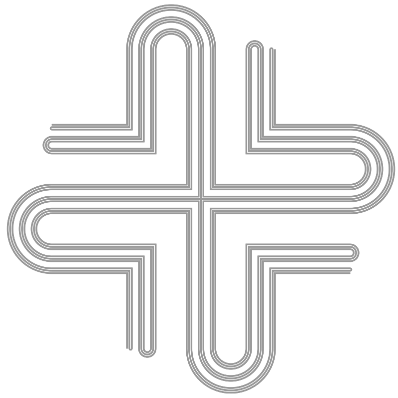


Fig. 2: Indecomposable continuum having four composants

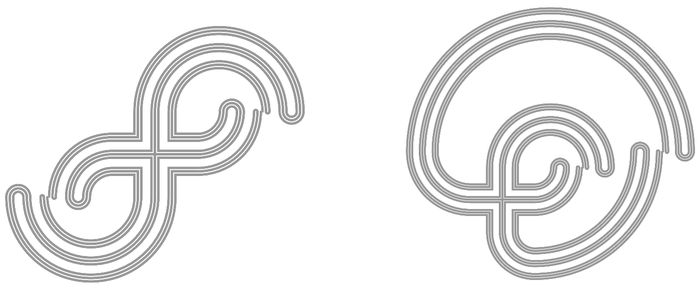


Fig. 3: There exist only two ways to glue the endpoints

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ON REVERSION OF THE ABEL–PRYM MAP AND ITS APPLICATIONS TO INTEGRABLE SYSTEMS

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Abel map transfoms a certain symmetric power of a Riemann surface to an Abelian variety called Jacobian of the Riemann surface. In the theory of integrable systems Abel map appeared as early as in K.Jacobi "Lectures on Dynamics". In frame of the method of Separation of Variables, the phase space of the system exfoliates into symmetric products of curves. If the curves are algebraic and equal, the Abel map transforms that foliation into the Lagrangian foliation of the system. It is wellknown that the trajectories of integrable systems are straight line windings of the Lagrangian tori (the fibers of the last foliation). To get trajectories explicitly, in the original separation coordinates, we need to reverse the Abel map. This problem is known as Jacobi inversion problem. Its solution is classical for Jacobians. However, for majority of classical and new integrable systems Lagrangian tori are not Jacobians but different Abelian varieties called Prym varieties, or Prymians. In general, no analog of Jacobi inversion can be formulated for Prymians. We hilight the case when the last nevertheless is possible. As application, we obtain solutions to the Hitchin system with the structure group SO(4) on a genus 2 curve in Prym theta functions (the most recent result is from 2002, by Krichever, in the GL(n) case; the previous results are due to Gawedzki and Tran-Ngoc-Bich'98, van Geemen and Previato'94 in the SL(2) case; to the best of our knowledge, no exact solution for orthogonal case was known). By means of general arguments we obtain a solution in genus 2 theta functions to the Kowalevski system. There is also a geometric outcome of our considerations, namely we represent the corresponding Prymians as symmetric powers of certain curves, up to birational equivalence. For an approach to the reversion of the Abel–Prym map in a different set-up see [2].

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INERTIA TENSOR OF A RIGID BODY IN PSEUDO-EUCLIDEAN SPACE

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The report will explore the connection between the following mechanical problems:

- the problem of the rotation of a rigid body around a fixed point (i.e., a spinning top) in a 3-dimensional (pseudo-)Euclidean space,
- the problem of the motion of a rigid body (referred to as a "plate") in a 2-dimensional space of constant curvature, specifically, on a 2-dimensional sphere, a Euclidean plane, or a Lobachevsky plane.

The inertia tensor of the rigid body from mechanics will be examined. We will describe its connection with the kinetic energy of the rigid body and the inertia tensor on the Lie algebras $so(3)$ and $so(2,1)$ in terms of a natural isomorphism between this Lie algebra and the ambient (pseudo-)Euclidean space. We will compute the inertia tensor of any single-point body in terms of the (pseudo-)Euclidean metric of the ambient space. As a consequence, firstly, it follows that the value of the inertia tensor (as a quadratic form) on any time-like vector is non-negative. In particular, the inertia tensor cannot have a signature of $(-,-,-)$. Secondly, for any "plate" on the Lobachevsky plane (lying within the light cone), the inertia tensor is positive definite. Thirdly, we will provide specific examples of two-point bodies lying outside the light cone, whose inertia tensor can have any signature other than $(-,-,-)$.

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ENERGY TRANSPORT AND CHAOS IN A ONE-DIMENSIONAL DISORDERED NONLINEAR STUB LATTICE

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We numerically study the dynamics of initially localized excitations in a one-dimensional stub lattice model in the presence of disorder and nonlinearity. The model's piece wise frequency spectrum is comprised by a near flat band and two non-flat spectra separated by distinct gaps when the disorder strength is below some threshold value. We theoretically predict and numerically observe three different dynamical regimes induced by chaos, namely the weak and strong chaos spreading regimes, and the self-trapping regime. Our numerical simulations show subdiffusive spreading for relatively large disorder strengths for both the weak and strong chaos regimes, which are characterized by specific exponents in the power law increase of the wave packets' second moment evolution in time. The system's chaoticity is quantified through numerical computations of the finite time maximum Lyapunov exponent, which is diminishing to zero following power law decays. Our findings show that the presence of frequency gaps does not have any significant effect on the wave packet spreading in the weak chaos regime, while they remain rather inconclusive for the strong chaos case, indicating the need for further investigations.

NEW TYPES OF SOLUTIONS TO THE VECTOR DERIVATIVE NLS EQUATION

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In our work we consider the following equations

$$\begin{aligned} &i\partial_z\mathbf{p} - \partial_t^2\mathbf{p} + 2i(\mathbf{p}^t\partial_t\mathbf{q})\mathbf{p} - 2(\mathbf{p}^t\mathbf{q})^2\mathbf{p} = 0,\\ &i\partial_z\mathbf{q} + \partial_t^2\mathbf{q} + 2i(\mathbf{q}^t\partial_t\mathbf{p})\mathbf{q} + 2(\mathbf{p}^t\mathbf{q})^2\mathbf{q} = 0, \end{aligned}$$

where $\mathbf{p}^t = (p_1, p_2, p_3)$, $\mathbf{q}^t = (q_1, q_2, q_3)$.

The corresponding Lax pair has the following form

$$i\Psi_t = U\Psi, \quad i\Psi_z = V_2\Psi,$$

where $U = -\lambda^2 J + \lambda Q + R$, $V_2 = \lambda^2 U + \lambda V_1^0 + V_2^0$,

$$J = \frac{1}{4}\begin{pmatrix} 3 & \mathbf{0}^t \\ \mathbf{0} & -I \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & \mathbf{p}^t \\ \mathbf{q} & \mathbf{0} \end{pmatrix}, \quad R = \begin{pmatrix} -\mathbf{p}^t\mathbf{q} & \mathbf{0}^t \\ \mathbf{0} & \mathbf{q}\mathbf{p}^t \end{pmatrix},$$

$$V_1^0 = \begin{pmatrix} 0 & \mathbf{H}_1^t \\ \mathbf{G}_1 & \mathbf{0} \end{pmatrix}, \quad V_2^0 = \begin{pmatrix} -f_1 & \mathbf{0}^t \\ \mathbf{0} & F_1 \end{pmatrix},$$

I is identity matrix, $\mathbf{H}_1 = -i\partial_t\mathbf{p}$, $\mathbf{G}_1 = -i\partial_t\mathbf{q}$,

$$F_1 = i\left(\partial_t\mathbf{q}\mathbf{p}^t - \mathbf{q}\partial_t\mathbf{p}^t\right) - \left(\mathbf{q}\mathbf{p}^t\right)^2, \quad f_1 = \mathrm{Tr}F_1.$$

In this case, the multiphase solutions correspond to non-hyperelliptic spectral curves $\Gamma = \{(\mu, \lambda)\}$ of the following form:

$$\mu^4 + A(\lambda)\mu^2 + B(\lambda)\mu + C(\lambda) = 0,$$

where

$$A(\lambda) = -\frac{3}{8}\lambda^{2n+4} + \sum_{k=1}^{n+2} A_k\lambda^{2n+4-2k}, \quad B(\lambda) = \frac{1}{8}\lambda^{3n+6} + \sum_{k\ge 1} B_k\lambda^{3n+6-2k},$$

$$C(\lambda) = -\frac{3}{256}\lambda^{4n+8} + \sum_{k\ge 1} C_k\lambda^{4n+8-2k}.$$

Since the genus of this curve is quite large already at $n = 1$, some solutions to the considered equation have unusual behavior.

To construct these solutions, we use spherical coordinates:

$$p_1 = |\mathbf{p}|e^{i\alpha_1} \sin\theta \cos\phi, \quad q_1 = \sigma p_1^*,$$
$$p_2 = |\mathbf{p}|e^{i\alpha_1} \sin\theta \sin\phi, \quad q_2 = \sigma p_2^*,$$
$$p_3 = |\mathbf{p}|e^{i\alpha_1} \cos\theta, \quad q_3 = \sigma p_3^*,$$

where $\sigma = \pm 1$.

In case $n = 1$, the length of the vector $\mathbf{p}$ is an elliptic function $u(t - kz)$, and the angles θ and ϕ depend on the length. For some parameter values, the direction of the vector $\mathbf{p}$ is fixed, only its length changes. However, there are also solutions when the vector $\mathbf{p}$ has a constant length, only its direction changes. Note that in the case of a two-component vector $\mathbf{p}$ (see [1]), there were no nontrivial solutions with constant length. Naturally, there are also solutions where the direction of the vector $\mathbf{p}$ does not trivially depend on its variable length.

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DECOMPOSITION OF LIE ALGEBRA INTO SUM OF TWO SUBALGEBRAS AND INTEGRABILITY

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Let $\mathfrak{g}$ be a Lie algebra with a basis $\mathbf{e}_i$, $i = 1, \ldots, n$. Suppose we have a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \tag{1}$$

where $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are subalgebras in $\mathfrak{g}$.

Let us consider the following non-linear system of ODEs

$$\frac{dU}{dt} = [\pi_+(U),\, U], \qquad U(0) = U_0. \tag{2}$$

Here

$$U(t) = \sum_1^n u_i(t)\, \mathbf{e}_i,$$

and π_+ denotes the projector onto $\mathfrak{g}_+$ parallel to $\mathfrak{g}_-$.

Proposition 1 (Adler-Kostant-Symes scheme). The solution of the Cauchy problem (2) is given by the formula

$$U(t) = A(t)\, U_0\, A^{-1}(t), \tag{3}$$

where the function $A(t)$ is defined as the solution of the following factorization problem:

$$A^{-1}\, B = \exp\left(-U_0\, t\right), \qquad A \in G_+, \quad B \in G_-. \tag{4}$$

Here G_+ and G_- are the Lie groups of the algebras $\mathfrak{g}_+$ and g_-, respectively.

The formula

$$[x, y]_R = 1/2\left([Rx, y] + [x, Ry]\right) \tag{5}$$

defines a second structure of Lie algebra on the vector space $\mathfrak{g}$. Here $R = \pi_+ - \pi_-$ is the difference of projectors on $\mathfrak{g}_+$ and $\mathfrak{g}_-$, respectively.

The operator R is the simplest example of the so called R-matrix. In general, the R-matrix is a linear operator $R : \mathfrak{g} \mapsto \mathfrak{g}$ that satisfy the modified Yang–Baxter equation

$$R\Big([y, R(x)] - [x, R(y)]\Big) + [R(x), R(y)] + [x, y] = 0,$$

where $x, y \in \mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{gl}_n$, $U = \sum u_{i,j}\mathbf{e_{ij}}$.

Lemma. Equation (2) is Hamiltonian with the Poisson bracket $\{\cdot, \cdot\}_R$, where $R = \pi_+ - \pi_-$, and Hamiltonian $H = \operatorname{trace} U^2$.

Despite the fact that the formula (3) gives an explicit solution of the equation modulo the factorization problem (4), the question of whether the equation is integrable in the classical sense and in particular whether it has the required number of integrals in involution in the general case remains open. In the case of decompositions

$$\mathfrak{gl}_n = \mathfrak{n}_+ \oplus \mathfrak{b}_-$$

and

$$\mathfrak{gl}_n = \mathfrak{so} \oplus \mathfrak{b}$$

the equation is bi-Hamiltonian, which ensures its integrability.

ON LOCALLY-FREE-COMPACTIFIED MODULI OF VECTOR BUNDLES ON A HIGHER-DIMENSIONAL VARIETY

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In my talk I will discuss the construction of the moduli scheme for semistable admissible pairs for arbitrary dimension. We start from a nonsingular projective algebraic variety S. The final result is an isomorphism between the moduli scheme of Gieseker-semistable torsion-free coherent sheaves of rank r and with Hilbert polynomial $rp(n)$ on the varity S with a fixed polarization L and the moduli scheme of semistable admissible pairs $((\widetilde{S}, \widetilde{L}), \widetilde{E})$. Each such pair consists of an admissible scheme $\widetilde{S}$ with a distinguished polarization $\widetilde{L}$ and a semistable locally free sheaf $\widetilde{E}$ of rank r and with Hilbert polynomial $rp(n)$. In particular, this provides a compactification of the moduli space of stable vector bundles by vector bundles on some special (admissible) schemes instead of the classical compactification by attaching non-locally free coherent sheaves. I will describe the notion of stability (semistability) for pairs $((\widetilde{S}, \widetilde{L}), \widetilde{E})$ ([1]) and develop a functorial approach to the construction of their moduli. This construction generalizes analogous results for the two-dimensional case ([2]).

The basement for the subject of interest is the Kobayashi–Hitchin correspondence. It allows one to apply algebraic geometrical methods to problems of differential geometrical or gauge theoretical setting by transferring consideration of the moduli of connections in a vector bundle (including vector bundles endowed with additional structures) to consideration of the moduli for vector bundles which are slope-stable. The compactification constructed is conceived as a tool for extending of the Kobayashi–Hitchin correspondence to the compact case.

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BYLLYARD DYNAMICS AS A SOURCE OF STONE'S LOGIC GENERALIZATION

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Quantum billiard is a dynamics system created by a particle which moves within a domain Ω and is reflected from the wall $\partial\Omega$ like a billiard ball. Wavefunctions ψ_n of this free particle are computed from Schroedinger's equation which is the spectral problem for Laplacian $-\Delta\psi_n = E_n\psi_n, \psi_n \mid_{\partial\Omega}= 0$. The eigenvalues E_n are proportional to admissible energy levels of the particle and form the spectrum of the billiard $Spec(\Omega) = \{0 < E_1 < E_2 \leq \ldots\}$.

In 1990s there was discovered a very challenging effect in mathematical physics. Namely, there were found first examples of **non-isometric** billiards $\Omega \not\approx \widetilde{\Omega}$ with the **same spectra** which coincide as countable sets $E_n = \widetilde{E}_n, n = 1, 2, \ldots$. [1] contains a whole gallery of such examples; a review of construction methods for these examples was made in [2]. This paper was the first step for the author in many years' development of general "isospectry method" which was recently finished (see complete proofs in [3]) and is summarized in the following statement:

Realization theorem [T., 2020]. *There exist only 2 realizations of isospectrality* $Spec(\Omega) = Spec(\widetilde{\Omega})$*:* **I)** *isometry* $\Omega \approx \widetilde{\Omega}$*;* **II)** *multi-valued isometry* $\mathbf{T}$*: then* Ω *and* $\widetilde{\Omega} = \mathbf{T}\Omega$ *are m-cellular with the same cell* Φ *and all cellular subdomains* $\omega \subset \Omega$ *generate new isospectral billiards:*

$$\omega = \mathbf{T}^{-1}\mathbf{T}\omega \Leftrightarrow Spec(\omega) = Spec(\mathbf{T}\omega) \quad (1)$$

Cellular billiards are constructed by mirror reflections of the same connected cell Φ over line segments on its boundary $\partial\Phi$. The case II covers all non-isometric billiards with the same energy spectra: both the known billiards and not yet found ones and excludes any other non-trivial ways to realize isospectrality. One may choose arbitrary subdomains $\alpha \subset \Omega$ to create continuum of new isospectral non-isometric billiards $\mathbf{T}^{-1}\mathbf{T}\alpha \subset \Omega$ and $\mathbf{T}\alpha \subset \widetilde{\Omega}$. 2-cellular isospectral billiards are domains with an axis of symmetry so they may be only isometric ones as the case II requires the same cell for them. This case confirms our approach as it agrees with spectral rigidity of axis-symmetrical domains proved in [4] using absolutely different approach.

(1) is valid for all cellular **subdomains** $\omega \subset \Omega$. Now we declare **any arbitrary subsets** $\omega \subset \Omega$ satisfying (1) as cellular and isospectral to $\mathbf{T}\omega \subset \widetilde{\Omega}$ by definition and denote the set of all such subsets as $\mathbf{S}$. All intersections, complements and unions of such subsets also belong to $\mathbf{S}$ due to the pure geometric condition for cellular subdomains $\omega = \mathbf{T}^{-1}\mathbf{T}\omega$ in (1). Then a topology emerges acc. to the following rule: a set ω is open if it is isospectral (1) to its image $\mathbf{T}\omega$, i.e. if they both are cellular ones. This **extremally disconnected** topology is well-known, $\mathbf{S}$ is the algebra of all its open-closed subsets with the unity Ω and plays the key role in the following principal Stone's theorem [5]:

Theorem [Stone, 1933] *Any complete Boolean algebra is isomorphic to Stone's algebra* $\mathbf{S}$ *of all open-closed subsets of extremally disconnected compact.*

Then subsets $\mathbf{T}\alpha \subset \widetilde{\Omega}$ may be treated as representations of subsets-statements $\alpha \subset \Omega$ and one can produce the table of **Stone's representation** of logical operations as operations with subsets $\mathbf{T}\alpha \subset \widetilde{\Omega}$:

statement: $\alpha \to \mathbf{T}\alpha$

conjunction: $\alpha_1 \wedge \alpha_2 \to \mathbf{T}\alpha_1 \cap \mathbf{T}\alpha_2$

disjunction: $\alpha_1 \vee \alpha_2 \to \mathbf{T}\alpha_1 \cup \mathbf{T}\alpha_2$

negation $\mathbf{A}$: $\mathbf{A}\alpha \to \widetilde{\Omega}\backslash\mathbf{T}\alpha$

Stone's representation gives logical correctness to the wave-particle duality of the quantum theory and presents models for dialectic transition of quantity into quality and unity and struggle of opposites. We calculate here Stone's representation for double negation $\mathbf{D}\alpha$ using the formula above for negation $\mathbf{A}\alpha$:

$$\mathbf{D}\alpha \to \Omega\backslash(\mathbf{T}^{-1}\mathbf{A}\alpha) = \mathbf{T}^{-1}\mathbf{T}\alpha \tag{2}$$

The equality $\alpha = \mathbf{D}\alpha = \mathbf{T}^{-1}\mathbf{T}\alpha$ means acc. to (1) that $\alpha \in \mathbf{S}$ and it is a cellular subset so generalized inclusion holds:

$$\alpha \subseteq \mathbf{D}\alpha, \tag{3}$$

where "=" corresponds to the law of excluded middle.

Another confirmation of our theory is the fact that the formula (3) is well-known in mathematical logics. Here is citation from a paper of a Moscow algebraist Kabakov F.A. (1972, his doctoral advisor was S.P.Novikov): "In the language of statement logics the double negation law is expressed with the formula

$$\neg\neg p \supset p \tag{4}$$

and is usually used in lists of **logical axioms**".

(3) and (4) are obviously the same with different notation $\alpha \leftrightarrow p, \mathbf{D} \leftrightarrow \neg\neg$, but (4) is an **axiom** and (3) on the contrary is a **strict consequence** derived from the quantum theory and Stone's theorem! So one can conclude that this Stone's logic generalization which looks so brightly in quantum theory is valid for all applications of Laplacian in other physical branches and promises fruitful perspectives.

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ASYMPTOTIC BEHAVIOR OF REGULAR AND IRREGULAR SOLUTIONS IN THE CAMASSA–HOLM EQUATION

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A periodic boundary value problem is considered for a modified Camassa–Holm equation, which differs from the well-known classical equation by several additional quadratic terms. Three important conditions on the coefficients of the equation are formulated under which the original equation has the Camassa–Holm type. The dynamic properties of regular solutions in neighborhoods of all equilibrium states are investigated. Special nonlinear boundary value problems are constructed to determine the "leading" components of solutions. Asymptotic formulas for the set of periodic solutions and finite-dimensional tori are obtained. The problem of infinite-dimensional tori is studied. It is shown that the normalized equation in this problem can be compactly written in the form of a partial differential equation only for the classical Camassa–Holm equation. An asymptotic analysis is presented in the cases when one of the coefficients in the linear part of the equation is sufficiently small, while the period in the boundary conditions is sufficiently large.

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LIOUVILLE FOLIATION OF INTEGRABLE BILLIARD BOOKS AT THE FOCAL LEVEL

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Let us fix a family of confocal quadrics by the relation

$$(b-\lambda)x^2 + (a-\lambda)y^2 = (b-\lambda)(a-\lambda).$$

Here a, b are fixed parameters of the family, which in particular fix the distance between the foci. If $a > b > 0$, this relation describes a family of confocal ellipses and hyperbolas, which include the focal line $y = 0$ and the limit hyperbola $x = 0$.

A billiard bounded by arcs of confocal quadrics is integrable [1]. For each trajectory, all its links tangent to the fixed ellipse or hyperbola from the same confocal family as the billiard boundary. Thus, the parameter Λ of this confocal quadric acts as an integral of the system. Let us separately select the focal level $\Lambda = b$ at which the links of the trajectories lie on the lines passing through the foci.

Let us consider the foliation of this integrable Hamiltonian system on a three-dimensional surface of constant energy. Such a foliation can be effectively described by Fomenko-Zieschang invariants [2], which was previously done by V. Dragovic, M. Radnovic and the speaker. However, if we generalize the class of integrable billiards by including so-called billiard books, then the class of different foliations (that is, the corresponding different invariants) is significantly expanded.

A billiard book is a two-dimensional CW-complex, the two-dimensional cells of which are parts of a plane bounded by arcs of confocal quadrics, and some cyclic permutations are assigned to the one-dimensional cells. A material point moving along a two-dimensional cell after reflection from a one-dimensional cell continues moving along the sheet to which the corresponding permutation points. By adding some natural conditions on the commutation of permutations in zero-dimensional cells, we obtain a completely integrable Hamiltonian system.

How can we describe the three-dimensional neighborhood of the focal level $\Lambda = b$ for a billiard book? It turns out that the following two results are true in this case.

For any non-degenerate three-dimensional atom (bifurcation of the foliation on the isoenergy surface of an arbitrary integrable Hamiltonian system) there exists a billiard book whose foliation on this layer coincides layerwise with the foliation of the given three-dimensional atom [3]. The trajectories of such a billiard book do not pass through the foci.

Theorem 1. *Let the trajectory of an arbitrary billiard book passes through the foci. Then the neigborhood of the connected part of the corresponding singular fiber is described by an atom belonging to one of the three series of symmetric atoms* X_n, Y_n, $A^{*\dots*}$ *(see the descriptions of the series in the book [2]).*

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PERIODIC INTERNAL TRANSITION LAYERS IN THE REACTION-DIFFUSION PROBLEM IN THE CASE OF A WEAK REACTION DISCONTINUITY

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One of the actual problems of the theory of singular perturbations is the study of nonlinear singularly perturbed partial differential equations, the solutions of which have boundary or internal layers. Such equations are of great interest both in the qualitative theory of differential equations and in many practical applications. In particular, in mathematical models of chemical kinetics, synergetics, nonlinear wave theory, biophysics and other fields of physics, where the processes under study are described by nonlinear parabolic equations with small parameters at derivatives. Solutions to such problems may contain narrow areas of fast parameter change: boundary or internal transition layers (contrast structures) of various types – stationary or moving fronts [1].

Reaction-diffusion and reaction-diffusion-advection equations are also intensively studied due to the fact that they act as mathematical models that reveal the main mechanisms that determine the behavior of more complex physical systems. In particular, the system of equations of the drift-diffusion model of a semiconductor with an N-shaped dependence of the drift velocity on the electric field strength can be reduced to the problem posed and considered below.

The reason for the transition layers (contrasting structures) appearance in singularly perturbed reaction-diffusion-advection models can be the fulfillment of the reaction balance condition at some point or on some curve lying in the field of consideration or advection balance, as well as the gap of coefficients by spatial coordinate [2–5].

In the works [2, 3] for the case of continuous coefficients, the so-called *critical case* was considered when the reaction balance condition is fulfilled identically, i.e. at any point in the domain. In this paper, we consider a periodic in time boundary value problem for a nonlinear singularly perturbed reaction-diffusion equation in the critical case in the presence of a weak discontinuity of the reactive term. By *weak discontinuity* is meant a discontinuity of the first kind at some point, which the function of sources undergoes in the first order with respect to a small parameter.

The following problem is considered:

$$\begin{cases} N_\varepsilon u := \varepsilon^2 \left(\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t} \right) - f(u,x,t) - \varepsilon f_1(u,x,t) = 0, \quad (x,t) \in D, \\ D = \{\, (x,t) \in \mathbb{R}^2 \ : x \in (-1,1),\ t \in \mathbb{R}\,\}, \\ f_1(u,x,t) := \begin{cases} f_1^{(+)}(u,x,t), & u \in I_u, \quad x > x_p, \quad t \in \mathbb{R}, \\ f_1^{(-)}(u,x,t), & u \in I_u, \quad x < x_p, \quad t \in \mathbb{R}, \end{cases} \\ \dfrac{\partial u}{\partial x}(-1,t,\varepsilon) = 0, \quad \dfrac{\partial u}{\partial x}(1,t,\varepsilon) = 0,\ t \in \mathbb{R}, \\ u(x,t,\varepsilon) = u(x,t+T,\varepsilon),\ x \in [-1,1],\ t \in \mathbb{R}. \end{cases} \tag{1}$$

Here $0 < \varepsilon < \varepsilon_0 \ll 1$ – small parameter, $x_p \in (-1;1)$, I_u - the change segment of the function $u(x,\varepsilon)$. Functions f, $f_1^{(\pm)}$ are quite smooth and T-periodic in t and $\lim\limits_{x \to x_p+0} f_1^{(+)}(u,x,t) \neq \lim\limits_{x \to x_p-0} f_1^{(-)}(u,x,t))$, $u \in I_u,\ t \in \mathbb{R}$.

For this singularly perturbed reaction-diffusion equation the solution with periodic in time internal transition layer is investigated in the case of a balanced reaction with a weak discontinuity. It is shown that in the case of the balanced reaction, the presence of even a weak (asymptotically small) reaction discontinue can lead to the formation of different finite size contrast structures, which may be stable or unstable.

The conditions under which there is a periodic in time solution of contrast structure type having an internal transition layer localized in the vicinity of the reaction break point are formulated. The existence of periodic solutions with an internal transition layer (contrast structures) is proved, the question of their stability is investigated, and an asymptotic approximation with respect to a small parameter of this solutions is constructed. Sufficient conditions are formulated that determine either the asymptotic Lyapunov stability or the instability of each such solution.

The asymptotic approximation is constructed according to the method [6]; the asymptotic method of differential inequalities is used to prove the existence of the solution [7], as well as the asymptotic method of differential inequalities developed for problems with discontinuous nonlinearities [4, 5]; the study of stability is carried out by the method of compressible barriers [2].

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RESURGENT NEURON MODE

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As a model of one neuron, the following equation is considered:

$$\dot{v} = \lambda(f_\beta(v(t-h)) + g(u^*_\lambda(t)))v, \tag{1}$$

where $v(t)$ is the normalized membrane potential, $h > 0$ is the time delay and $\lambda \gg 1$ is the speed of electrical processes in the nerve cell. Function $f_\beta(v),\ g(u) \in C^\infty$ satisfy the conditions

$$f_\beta(0) = 1, \quad \lim_{u\to+\infty} f_\beta(u) = -\beta, \quad g(0) = -\eta, \quad \lim_{u\to+\infty} g(u) = \xi.$$

The function $u^*_\lambda(t) = e^{\lambda x^*_\lambda(t)}$ is T^*_λ-periodic and satisfies the conditions:

$$\lim_{\lambda\to+\infty} \max_t |x^*_\lambda(t) - x^*(t)| = 0, \ \lim_{\lambda\to+\infty} T^*_\lambda = T^*,$$

$$x^*(t) \stackrel{\text{def}}{=} \begin{cases} t, & t \in [0,1], \\ -\alpha(t-t^*), & t \in [1, t^*+1], \\ t - T^*, & t \in [t^*+1, T^*], \end{cases}$$

$$x^*(t+T^*) = x^*(t), \quad t^* = (\alpha+1)/\alpha, \quad T^* = (\alpha+1)^2/\alpha.$$

Here, β, η, ξ, α are positive parameters.

Equation (1) is a modification of the equation

$$\dot{u} = \lambda f(u(t-h))u, \tag{2}$$

proposed in the article [2]. Where $u = u(t) \geq 0$, $\lambda \gg 1$, function $f(x)$ is infinitely differentiable on $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ such that $f(0) = 1$, $f(x) \to -\alpha$ as $x \to +\infty$. This equation underlies a number of phenomenological neuromodels.

In the work [1], the existence of solutions close to the damped neuro oscillator mode for equation (1) was analytically proved. Solutions of this type initially exhibit any predetermined number of exponentially high spikes, followed by a gradual attenuation of the spikes and the establishment of exponentially small oscillations.

In this work, the results of numerical experiments are presented, during which a new regime of neuro-oscillator behavior, termed the resurgent regime, was discovered. The essence of the resurgent regime is as follows. Over an interval proportional to the large parameter, the solution is asymptotically close to that of the fading neuro-oscillator type. Subsequently, the solution becomes close to a periodic one with a bursting effect.

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SYNCHRONIZATION OF SLOWLY OSCILLATING SOLUTIONS IN A SYSTEM OF COUPLED EQUATIONS WITH NEUTRAL DELAY

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Consider a system of differential equations of neutral type

$$\begin{cases} \dot{x} + x = \alpha \cdot \operatorname{sign}(\dot{x}(t-T)) + \gamma(y-x), \\ \dot{y} + y = \alpha \cdot \operatorname{sign}(\dot{y}(t-T)) + \gamma(x-y), \end{cases} \tag{1}$$

where $\alpha < 0,\ T > 0, \gamma \geqslant 0$, with the initial condition

$$\begin{cases} \dot{x}(t) > 0, \quad t \in [-T, 0], \quad x(0) = h_1, \\ \dot{y}(t) > 0, \quad t \in [-T, 0], \quad y(0) = h_2. \end{cases} \tag{2}$$

For $\gamma = 0$, system (1) consists of two independent differential equations of the same form, the solutions of which were studied in [1].

Theorem 1. *For $h_1 = h_2 = h^* = \alpha \dfrac{1-e^T}{1+e^T} \in (-|\alpha|, |\alpha|)$ solution of system (1) with initial condition (2) is synchronized (i.e. $x(t) = y(t)$) and periodic, and if*

$$\begin{cases} h_1, h_2 \in (-|\alpha|, |\alpha|), \\ h_1 < h_2, \\ \min(h_1, h_2) > \dfrac{\alpha + \gamma \max(h_1, h_2)}{1+\gamma}, \\ \max(h_1, h_2) < \dfrac{\min(h_1, h_2)(1 + 2\gamma - e^{2\gamma T}) - 2\alpha e^{2\gamma T}}{1 + 2\gamma + e^{2\gamma T}}. \end{cases}$$

then corresponding solution approaches this synchronized and periodic solution.

Now consider the initial conditions

$$\begin{cases} \dot{x} > 0 \text{ at } t \in [-T, -\theta T], \\ \dot{x} < 0 \text{ at } t \in [-\theta T, 0], \\ \dot{y} > 0 \text{ at } t \in [-T, 0], \\ x(0) = h_1,\ y(0) = h_2, \end{cases} \tag{3}$$

where $\theta \in [0, 1]$ is the desynchronization parameter. The case $\theta = 0$ corresponds to the case of synchronization $(x(t) = y(t))$, the case $\theta = 1$ corresponds to the case of antisynchronization $(x(t) = -y(t))$.

Theorem 2. *There are unique h_1^*, h_2^* such that if $h_1 = h_1^*$, $h_2 = h_2^*$ and*

$$e^{-T}\left(\frac{h_1 + h_2}{2} - \alpha\right) - e^{-T(1+2\gamma)}(1+2\gamma)\frac{h_1 - h_2}{2} - \alpha e^{-T(1+2\gamma)\theta} + \alpha e^{-T\theta} > 0, \tag{4}$$

then the solution to equation (1) with the initial condition (3) is periodic. Moreover, this periodic solution is stable, and for the initial condition (3) with h_1 and h_2 sufficiently close to h_1^* and h_2^*, all solutions tend exponentially to this periodic solution.

From the analysis of inequality (4) it follows that for small T and γ desynchronized periodic solutions coexist for any values of desynchronization θ; As T and γ increase, some modes become impossible, and only modes close to synchronized or antisynchronized remain.

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DELAY-INDUCED RADIATION OSCILLATIONS IN CHAIN OF A LARGE NUMBER OF PUMP-COUPLED LASERS

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Radiation dynamics of a closed chain of lasers with optoelectronic delayed coupling is analyzed. We consider a chain with unidirectional coupling, bidirectional coupling without feedback, diffusion-like coupling. Assuming that the number of lasers is sufficiently large, we propose the phenomenological spatially distributed models. The coupling level is determined at which the stationary state of laser generation becomes unstable. A two-dimensional complex partial differential equation of the Ginzburg-Landau type is derived as a quasi-normal form. Based on its simplest solution we describe radiation oscillations which can be phase synchronized, anti-phase or in-phase in dependence on time delay.

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