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First Passage Time Probabilities in Diffusion Models

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This work studies the hitting time problem for time-dependent diffusion processes involving the crossing of single and double barriers, where the boundaries may be time-dependent. The approach is based on the local time-space formula of Peskir and a Markov chain approximation of the transition density function. We assume a diffusion process X governed by a stochastic differential equation (SDE) on $\mathbb{R}$ of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x, \tag{1}$$

where $\mu : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is the drift term, $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is the diffusion, and W is the standard Brownian motion under the measure $\mathbb{P}$. We assume that the SDE (1) admits a unique strong solution with a transition probability $\mathbb{P}_{t,x}(X_s \in A) = \mathbb{P}(X_s \in A \mid X_t = x)$, defined for $0 \le t \le s$ and any Borel set $A \subset \mathbb{R}$. A smooth transition density function (tdf) $p(s, y; t, x)$ determined by the relation $\mathbb{P}(X_s \in A \mid X_t = x) = \int_A p(s, y; t, x)dy$. We define the first hitting time for the lower barrier $b(t)$ in C^1 as

$$\tau_b = \inf\{t \ge 0 : X_t \le b(t)\} \tag{2}$$

with $X_0 = x > b(0)$. The main result is semi-analitical expression for the cumulative hitting probability $G(t, x) := \mathbb{P}_x(\tau_b \le t)$. The proposition for time-homogeneous case is following

Теорема 1. *The cumulative probability function $G(t, x)$ of the hitting time is given as*

$$G(t, x) = \mathbb{P}_{0,x}(X_t \le b(0)) + \int_0^t f(u)p(t, b(u); u, x)du \tag{3}$$

for $t \in [0, T]$ and $x \in \mathbb{R}$, where $f(u)$ is the solution of the linear Volterra equation of the first kind

$$\mathbb{P}_{0,b(t)}(X_t > b(0)) = \int_0^t f(u)p(t, b(u); u, b(t))du \tag{4}$$

for $t \in (0, T]$.

A similar type of result can be obtained for the first hitting time of the upper barrier $b(t)$ with $X_0 < b(0)$.

Since the formula above depends on $p(s, y; t, x)$, we propose an effective algorithm for the numerical computation of the tdf based on a Markov Chain approximation. The key idea is to approximate the process X by a continuous-time Markov chain with a finite state space $\{x_1, x_2, \dots, x_n\}$ on a uniform lattice, for which the transition probability matrix between two arbitrary time points can be computed explicitly. The problem is examined in the context of financial applications. An algorithm is presented, together with numerical results.