Секция «Прикладная математика и информатика»

Lattice Path Enumeration: Forbidden Zones and a Derivation of the Catalan Numbers

Саргсян Мари Енок
E-mail: marisargsyan.1644@gmail.com

<p>Lattice path problems are one of the most fundamental parts of discrete mathematics,as these types of problems stand as the most applicable problems in combinatorics.With this work my aim is to examine an example of lattice path enumaration problem.

In a nxn integer grid,how many monotone pathways can be formed from the (0,0)to the (n,n) point if only rightward and upward unit steps can be used?Make sure that the path does not ascend the main diagonal.This last requirement means that the number of upward steps never exceeds rightward steps.The problem states that the region above the diagonal is a forbidden zone, so the paths we find that violate this contraint should be excluded from the count.Here also should be mentioned that this formulation connects to the Catalan numbers, which is one of the most well-known sequences in combinatorics.

The first step into solving the problem and counting the valid paths should be observing the total number of unrestricted monotone paths from the starting point to the destination (n, n) point. Since each path consists of exactly 2n steps,nof which is are righward and n are upward,the binomial coefficient is C(2n, n).Now the next step should be isolating the paths that stay on or below the the diagonal.For that, we need to apply Andre's reflection principle.The concept is that every invalid path can be uniquely mapped to a map from (-1,1) to (n,n) by reflecting the portion of the path before its first violation across line y=x+1.With this,we establish a one-to-one correspondance between invalid paths and all paths from (-1,1) to our destination. The count of those paths becomes C(2n, n-1).Subtracking, we get that the number of valid paths equals C(2n,n)-C(2n, n-1), which simplifies the closed form C(2n, n)/(n+1). This quantity is precisely the n-th Catalan number:Cn.
The n-th Catalan number, given by Cn=C(2n,n)/(n+1), counts exactly the valid paths-for instance,yielding 5 paths for n=3-and represents only a 1/(n+1) traction of all enrestricted routes, despite growing exponentially;as Stanley (2011,p.222) notes, Catalan numbers count lattice paths 'never going above the diagonal.'

Constrained lattice path problems' usage goes beyond pure mathematics, and it has direct applications in computer science and technology.Underlying routing algorithms in chip design,for example , or their usage in probabilistic models in machine learning makes understanding these problems a foundational skill in today's technological world.

References
Koshy, T.(2008).Catalan Numbers with Applications.Oxford University Press.
Stanley,R.P.(2011).Enumerative Combinatorics,Volume 1(2nd ed.). Cambridge University Press.
Weisstein,E.W.(n.d).Catalan Number.MathWorld-A Wolfram Web Resource.<

Иллюстрации

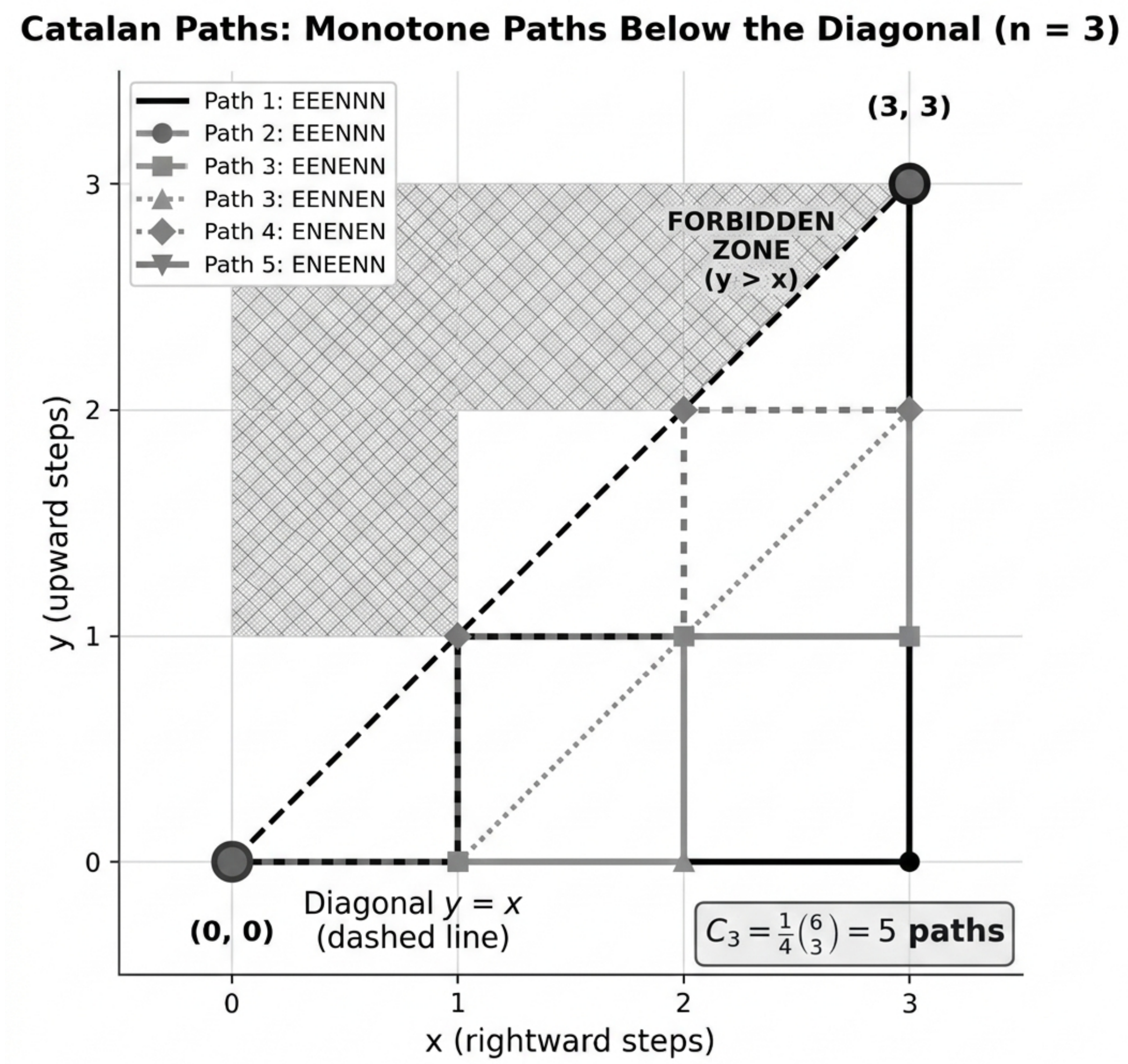
Catalan Paths: Monotone Paths Below the Diagonal (n = 3)
Path 1: EEENNN
Path 2: EEENNN
Path 3: EENENN
Path 3: EENNEN
Path 4: ENENEN
Path 5: ENEENN
(3, 3)
FORBIDDEN ZONE (y > x)
y (upward steps)
Diagonal y = x (dashed line)
(0, 0)
C₃ = 1/4 (6 3) = 5 paths
x (rightward steps)
0
1
2
3

Рис.