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A comparison theorem for a class of Riccati differential equations and its application to a pollution control problem

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We define the class $\mathcal{P}$ by saying that $\Pi_t^{(i)} \in \mathcal{P}$ if $\Pi_t^{(i)}$, $t \geq 0$, is a bounded absolutely continuous function with values in the set of symmetric positive semidefinite $n \times n$ matrices, which satisfies the Riccati equation

$$\dot{\Pi}_t^{(i)} = -\Pi_t^{(i)}(A_t + \psi_t^{(i)} \cdot I) - (A_t' + \psi_t^{(i)} \cdot I)\Pi_t^{(i)} + \Pi_t^{(i)} M_t \Pi_t^{(i)} - Q_t\,, \tag{1}$$

where A_t, Q_t, M_t are bounded time-varying matrices, $Q_t \geq 0$, $M_t \geq 0$ are symmetric, $\psi_t^{(i)}$ is a bounded function; $'$ denotes the matrix transpose, here the notation $X \geq Y$ means that the difference $X - Y$ is positive semidefinite, I is an identity matrix.

According to a well-known comparison theorem (see, e.g., [1, Theorem 4.1.4, p.185]) for solutions of Riccati differential equations, $\Pi_t^{(1)} \leq \Pi_t^{(2)}$, $t_0 \leq t$, if, along with $\Pi_{t_0}^{(1)} \leq \Pi_{t_0}^{(2)}$, the matrix $\begin{pmatrix} 0 & h_t \cdot I \\ h_t \cdot I & 0 \end{pmatrix}$ is positive semidefinite with $h_t = \psi_t^{(2)} - \psi_t^{(1)}$. Clearly, this condition is not fulfilled for $\Pi_t^{(i)} \in \mathcal{P}$.

Set $A_t^{(i)} = A_t + \psi_t^{(i)} \cdot I$. Below we use a common condition to ensure existence of a solution $\Pi_t^{(i)} \in \mathcal{P}$, namely that of stabilizability of the pair $(A_t^{(i)}, \sqrt{M_t})$.

Definition. [2, p.14] The pair $(A_t^{(i)}, \sqrt{M_t})$ is called stabilizable if there exists a bounded piecewise continuous matrix K_t such that the matrix $A_t^{(i)} + \sqrt{M_t} K_t$ is exponentially stable.

The main result is the following

Theorem 1. Let the pairs $(A_t^{(i)}, \sqrt{M_t})$ be stabilizable $(i = 1, 2)$. If $\phi_t^{(1)} \leq \phi_t^{(2)}$, $t \geq 0$, then

$$\Pi_t^{(1)} \leq \Pi_t^{(2)}\,, \quad t \geq 0\,.$$

Based on this result, one can obtain a comparison theorem for $\tilde{\Pi}_t^{(i)}$ satisfying

$$\dot{\tilde{\Pi}}_t = -\tilde{\Pi}_t A_t - A_t'\tilde{\Pi}_t + \tilde{\Pi}_t \frac{M_t}{g_t} \tilde{\Pi}_t - g_t Q_t\,, \tag{2}$$

where $g_t > 0$ is a function with bounded logarithmic derivative $\dot{g}_t/g_t$, $t \geq 0$.

Setting $\Pi_t = \tilde{\Pi}_t/g_t$, we immediately get the following

Corollary 1. Let the pairs $(A_t^{(i)}, \sqrt{M_t})$ be stabilizable $(i = 1, 2)$ with $\psi_t^{(i)} = (1/2) \cdot \dot{g}_t/g_t\,.$ If $g_t^{(1)} \leq g_t^{(2)}$, $t \geq 0$, then

$$\frac{g_t^{(2)}}{g_t^{(1)}} \cdot \tilde{\Pi}_t^{(1)} \leq \tilde{\Pi}_t^{(2)}\,, \quad t \geq 0\,. \tag{3}$$

Remark. The corresponding solutions $\tilde{\Pi}_t^{(i)}$ may possess specific asymptotic properties such as $\|\tilde{\Pi}_t^{(i)}\| \to 0$, if $g_t^{(i)} \to 0$, $t \to \infty$, or $\|\tilde{\Pi}_t^{(i)}\| \to \infty$ for $g_t^{(i)} \to \infty$, $t \to \infty$ ($\|\cdot\|$ denotes the Euclidean matrix norm).

Next we apply Theorem 1 to investigate the behaviour of optimal trajectories in a pollution control problem.

Suppose that the dynamics of stock pollutant S_t is given by

$$dS_t = -aS_t dt + Z_t dt\,, \qquad S_0 = s_0, \tag{4}$$

where S_t is the stock at time t; constant $a > 0$ represents the absorption capacity; s_0 – initial stock; the control $\{Z_t\}_{t=0}^{\infty}$ is the level of emissions at time t, i.e. a piecewise continuous function; denote by $\mathcal{Z}$ the set of admissible controls.

The cost functional over the planning horizon $[0, T]$ is defined by

$$J_T(Z) = \int_0^T f_t \cdot [q(S_t - \bar{S})^2 + q_1(Z_t - \bar{Z})^2]\, dt, \tag{5}$$

with some constants $q, q_1 > 0$ and $\bar{S}$, $\bar{Z}$ as the target levels of stock and emission. Note that we require $\bar{Z} = a\bar{S}$ for the stabilization purpose. The discount function $f_t > 0,\ f_0 = 1$, represents time preference of a decision-maker. Recall that f_t increases (decreases) for negative (positive) time-preference, it is constant when time preference is zero; we assume that the discount rate $\phi_t = -\dot{f}_t/f_t$ is bounded.

The control problem is

$$\limsup_{T\to\infty} J_T(Z) \to \inf_{Z\in\mathcal{Z}}\,.$$

The optimal emission policy Z^* has the form

$$Z_t^* = -\frac{1}{q_1}\Pi_t(S_t^* - \bar{S}) + \bar{Z}\,, \tag{6}$$

where the optimal stock $S_t^* = \bar{S} + s_0 \exp\{-at - (1/q_1)\int\limits_0^t \Pi_s\, ds\}$, and $\Pi_t > 0,\ \ t \geq 0$, satisfies

$$\dot{\Pi}_t - 2a\Pi_t - \phi_t\Pi_t - \frac{\Pi_t}{q_1} + q = 0\,. \tag{7}$$

To distinguish different types of time preference, we use the notations $f_t^{(+)}$, $f_t^{(0)}$ and $f_t^{(-)}$ for discount functions reflecting positive, zero and negative time preference, respectively. The corresponding optimal stocks are denoted by $S_t^{*(+)}$, $S_t^{*(0)}$, $S_t^{*(-)}$. Then we have

Corollary 2.

$$0 \leq S_t^{*(-)} - \bar{S} \leq S_t^{*(0)} - \bar{S} \leq S_t^{*(+)} - \bar{S}, \quad t \geq 0\,. \tag{8}$$

The interpretation of this result seems obvious. The more important is future, the less deviation of the stock is resulted from use of the optimal control law Z_t^*.

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Литература

1. Abou-Kandil H., Freiling G., Ionescu V., Jank G. Matrix Riccati Equations in Control and Systems Theory. Basel: Burkhauser Verlag, 2003.

2. Ichikawa A., Katayama H. Linear Time Varying Systems and Sampled-data Systems. London: Springer, 2001.